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A Non-Iterative Approximation for Critical Values for the Dunnett's Test

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Abstract

The Dunnett's test is sometimes used to discern if means of treatment groups are different than a control group. The method relies on the availability of quantiles from a distribution that is not available on calculators or in spreadsheet programs. Thus, a new non-iterative method was developed that can be implemented in spreadsheets, which are often used for data storage and evaluation. The method was incorporated as a Microsoft Excel user-defined function which requires Microsoft Visual Basic for Applications (VBA). In addition, the method was implemented as simple spreadsheet formulas that does not require VBA. The method provides values of the quantiles with low error rates compared with exact iterative methods. The mean absolute error rates of the developed method were below 0.3% for a variety of conditions and levels of statistical significance.

1. INTRODUCTION

Researchers in many fields such as biological, agricultural, environmental, and medical often engage in comparisons of data means obtained under various treatment conditions. Multiple comparison procedures (MCP's) or Pairwise Multiple Comparisons (PMC) are often carried out by statistical software packages that may also require the user to learn a new computer programming skill [1] and may be expensive [2]. Groups receiving different treatments and the comparison of those means is a post-hoc analysis that sometimes takes place after a general analysis of variance (ANOVA) evaluation. The methods by which means are compared differ depending on factors such as planned versus unplanned comparisons, comparisons between groups of the same or different sizes, comparisons of groups with or without equal variances, and simultaneous tests versus stepwise comparisons. A comprehensive review of different MCP methods may be found in the literature [3-5]. When pairwise comparisons are needed for all groups, Day and Quinn [3] recommends Ryan’s Q method [a.k.a. Ryan-Einot-Gabriel-Welsch Q (REGWQ)] when group variances are equal and Games-Howell’s method when group variances are not equal. In the case of comparisons of treatment with a control the Dunnett test was recommended. The two first test methods require Studentized range (Q) distribution and the Dunnett test require Dunnett’s t_D distribution. In contrast, Fisher LSD, Scheffe, Bonferroni, and Dunn-Sidak methods require the F distribution or Student t distribution. The F and t distributions are readily available in spreadsheet programs such as Microsoft Excel (Redmond, WA) and Google Sheets (Mountain View, CA) (with FINV and TINV functions), while the Q distribution is only available as an Excel Visual Basic module or add-in [6-8] or as part of statistical programs such as RStudio (Boston, MA) [using qtukey(…..) statement] and SAS Studio (SAS Institute, Cary, NC) [using probmc(“RANGE”,…..) statement]. However, values from the Dunnett’s t_D distribution is not available in spreadsheets or calculators and must be retrieved from tables [9], complex algorithms [10], or statistical programs such as RStudio [using qNCDun(…..) statement with the nCDunnett library] or SAS Studio [using probmc(“DUNNETT2”,…..) statement]. As these statistical programs may not be available to all and tables may be limited to some α’s, I am proposing an alternative method for calculating the approximate t_D values to those found in by the methods above. Microsoft Excel was used as a base platform but, as I show, other spreadsheet programs may also be used.

2. THEORY AND AN EXAMPLE

As explained by Dunnett [9], the testing of several treatments against a common control (or common treatment) and the motivation for creating t_D tables was based on the desire to control the experimental Type I error rate rather than the per comparison error rate. The method was rather simple in its structure and was recommended when several treatment options are to be compared with a control case, but comparisons are not made between treatments. The method is based on calculating Student t statistics as [9]:

$$t = | \bar{x}_t - \bar{x}_c | \times [MS_w \times (1/n_t + 1/n_c)]^{-0.5}, \qquad (1)$$

where, $\bar{x}_t$ and $\bar{x}_c$ are the means of the treatment and control groups, n_t and n_c are the number of observations in the treatment and control groups, and MS_w is the mean squares within groups (from an ANOVA Table). If this t value is greater than the t_D value from tables [9], then there is a difference between the treatment and control means. To illustrate, we can use the example by Dunnett [11] where the data represent breakage measurements of fabric treated by three chemical processes. In the example, triplicate experiments were conducted with the result as follows: 55, 47, and 48 for the control; 55, 64, and 64 for Process 1; 55, 49, and 52 for Process 2; and 50, 44, and 41 for Process 3. While not necessary, one-way ANOVA evaluation yields that the null hypothesis is not true and that means are not the same at the $\alpha = 0.05$ confidence level, with MS_w and df_w equal to 19 and 8, respectively.

The t statistics for Process 1, 2, and 3 can be calculated using Equation 1 to be 3.09, 0.56, and 1.40. When compared to the tabulated value of 2.88 [9] for $\alpha = 0.05$, $g = 4$ groups including control, and $df_w = 8$, it is concluded that only Process 1 is statistically different than the control. The Dunnett's test may be rewritten in a form of a general statement for testing the null hypothesis (that the means of two groups are equal) with a general safety factor (SF). If the difference between group means (say groups a and b) is less than a Critical Value (CV),

$$CV = SF \times [\ MS_w \times (\ 1/n_a + 1/n_b\ )\ ]^{0.5}\ , \quad (2)$$

then there is no difference between the two groups. In case of the Dunnett's test (where α could be the control group and b the treatment group). The SF for Dunnett's and a few other multiple pairwise test method is shown below:

$$\text{Dunnett} \qquad SF = t_D(\alpha,g,df_w)\ , \quad (3)$$

$$\text{Fisher LSD} \qquad SF = [\ F(\alpha,1,df_w)\ ]^{0.5}\ , \text{and} \quad (4)$$

$$\text{Tukey HSD} \qquad SF = 0.5^{0.5} \times Q(\alpha,g,df_w)\ . \quad (5)$$

With $\alpha = 0.05$, $g = 4$ (groups including control), and $df_w = 8$, the Safety Factor for each test is 2.88 for Dunnett's test (from Table II by Dunnett [9]), 2.31 for Fisher's LSD test (using the FINV Excel function), and 3.20 for the Tukey HSD test (using the QTAB or QINVG Excel functions [6, 7]. Noticeably, the only test where tables must be consulted is for the Dunnett's test.

3. DEVELOPMENT OF ALGORITHMS TO CALCULATE t_D VALUES

The methods described by Dunnett [9] and Kwong and Liu [10] to calculate t_D are very computationally intensive, and it would be useful to have an alternative. As Dunnett [9] was inspired by the work of Tukey on controlling the experimental error rate with Q values (Studentized range quantiles), I followed the same non-iterative approach which Gleason [12] proposed for Q estimations, i.e., assume that quantiles are functions of α, g, and v (i.e., df_w), and can be expressed as a polynomial function of ln(g – 1) by

$$[\ t_D(\alpha,g,v) / t(\alpha,v)\ ] - 1 = A_1(\alpha,v) \times \ln(g-1) + A_2(\alpha,v) \times \ln^2(g-1)\ . \quad (6)$$

The relationship works very well as seen in Figure 1. Note that there would be a unique polynomial for each combination of α and v. The approach by Gleason [12] used a 4th order polynomial correlating $Q(\alpha,g,v) / t(\alpha,v)$ with ln(g – 1). However, in this case, higher than 2nd order polynomial did not significantly improve the fit.

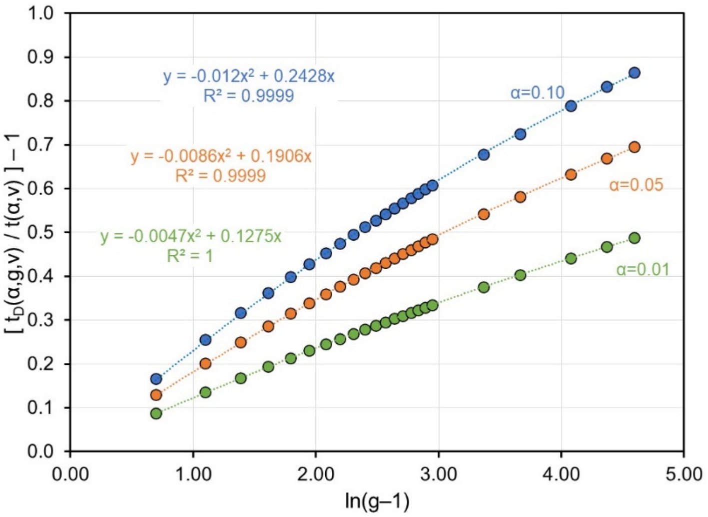


Figure 1. Correlation expressed by Equation 6 for three different α values with v = 120 and g between 3 and 100.

The determination of the values of A_1 and A_2 for large selection of α and v was done using multiple variable regression. This resulted in a table as shown in the Supplementary Materials (Table S1). The correlations already allow for interpolation in g. The correlations can be extended for interpolation in v and in α, using the same principles as suggested by Gleason [12] for Q values. Considering v, the interpolation of t_D for an unlisted v can be done, using the two v values that bracket the desired v value:

Calculate a y^2 value for each of the v values (bracketing the desired v value) from the following equation, where g is known and A's are available from Equation 6 correlations (see Table S1):

$$y^2 = [\, t_D(\alpha,g,v) / t(\alpha,v) \,]^2 = [\, 1 + A_1(\alpha,v) \times \ln(g-1) + A_2(\alpha,v) \times \ln^2(g-1) \,]^2 \tag{7}$$

This will result in two y^2 values. Now, assume that y^2 is linearly depended on 1/v and interpolate y (via y^2) with 1/v as follows:

$$y = [\, y^2_{low} + (1/v - 1/v_{low}) \times (y^2_{high} - y^2_{low}) / (1/v_{high} - 1/v_{low}) \,]^{0.5} \tag{8}$$

Calculate the t_D value as follows:

$$t_D(\alpha,g,v) = y \times t(\alpha,v) \tag{9}$$

When considering the interpolation of t_D with an unlisted α (at a specified g and v), the following procedure can be followed.

For each of the eight available α values, calculate a y value as

$$y = 1 + A_1(\alpha,v) \times \ln(g-1) + A_2(\alpha,v) \times \ln^2(g-1) \tag{10}$$

This will yield eight y values, one for each α. Assume that the y value depends on the α value through the following equation, introducing the z variable:

$$z = \ln(y + g/v) = b_1 + b_2 \times x + b_3 \times x^2 \tag{11}$$

Where x is function of the standard normal distribution in the following form:

$$x = -[\, 1 + 1.5 \times U(1 - \alpha/2) \,]^{-1} \tag{12}$$

and U is the abscissa value where the cumulative area under the standard normal distribution ($\mu = 0$ and $\sigma = 1$) curve is $1 - \alpha/2$. This value can be calculated by the NORMINV Excel function.

The relationship between x and z can be seen in Figure 2 for selected v values. As is noted, the fit to a second-order polynomial (as in Equation 11) is excellent. Thus, from the eight α values, we get eight z values and eight x values. The parameters b_1, b_2, and b_3 in Equation 11 can now be determined through multiple variable regression.

The final value of t_D at the desired α can now be calculated by calculating its x value by Equation 12, then its z value (and by extension its y value) via Equation 11, and finally its t_D value by Equation 9.

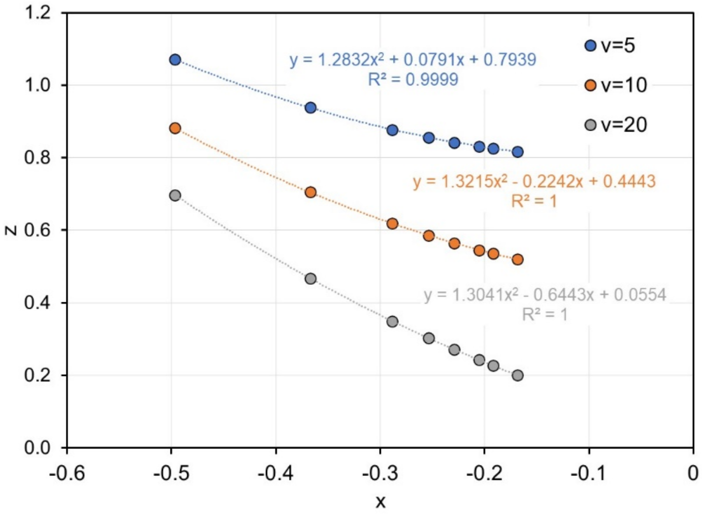


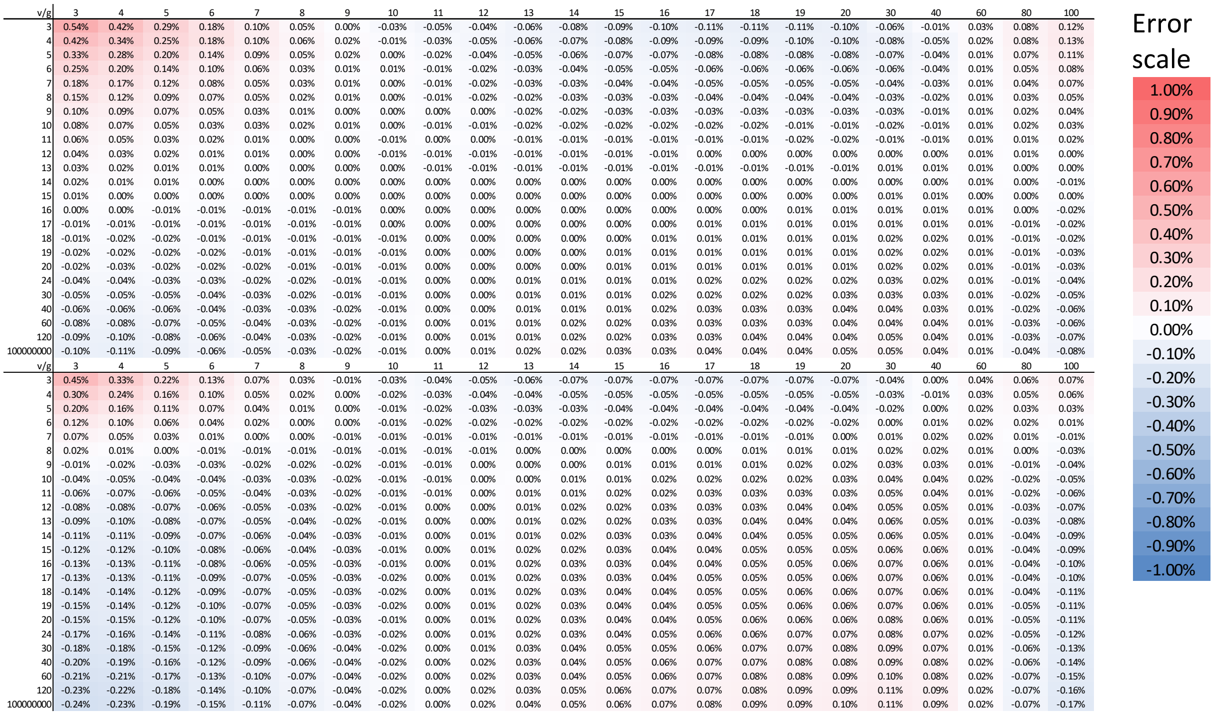
Figure 2. Demonstration of the fit between x and z, according to Equation 11. Each point corresponds to an α value (0.001 to 0.5). The number of groups (g) was set to 5.

4. APPLICATIONS

To demonstrate its application, the above procedure was incorporated as a user-defined function in Excel which was named TDINV to compliment other built-in Excel functions, such as FINV and TINV. The TDINV function takes three arguments, namely α, g, and v. The VBA code can be seen in the Supplementary Materials (Code S1) and should be incorporated as a VBA Module in each Excel Workbook where it will be used, or it can be used as an Excel add-in, for availability to all workbooks on the same computer. The method developed is ideal for Excel as it has the Visual Basic for Applications feature. However, this is not an option for other spreadsheet application such as Google Sheets. The same methodology can be used by devoting part of a workbook to the A tables together with interpolation equations. Such a spreadsheet is also included in the Supplementary Materials (Figure S1). While created in Excel, it can directly be loaded into Google Sheets. While this method works, it does require some space in the workbook or maybe as a separate worksheet. The space could be reduced if only a few of the most used α values are considered (e.g., 0.1, 0.05, and 0.01).

5. RESULTS AND DISCUSSION

The error map of t_D values calculated using the created TDINV function compared with those obtained by SAS Studio [using probmc("DUNNETT2",.....) statement] are shown in Figure 3 for α = 0.01 and 0.05, where dark blue and dark red represent greater negative and positive error, respectively. Other error maps for α = 0.50, 0.25, 010, 0.025, 0.005, and 0.001 are included in the Supplementary Materials (Figures S2-S7). A summary of the highest, lowest, and mean absolute errors for different values of α are shown in Table 1. As is noted, the errors in determining t_D values via the proposed method are low; on average 0.03-0.08% for α ≤ 0.1.



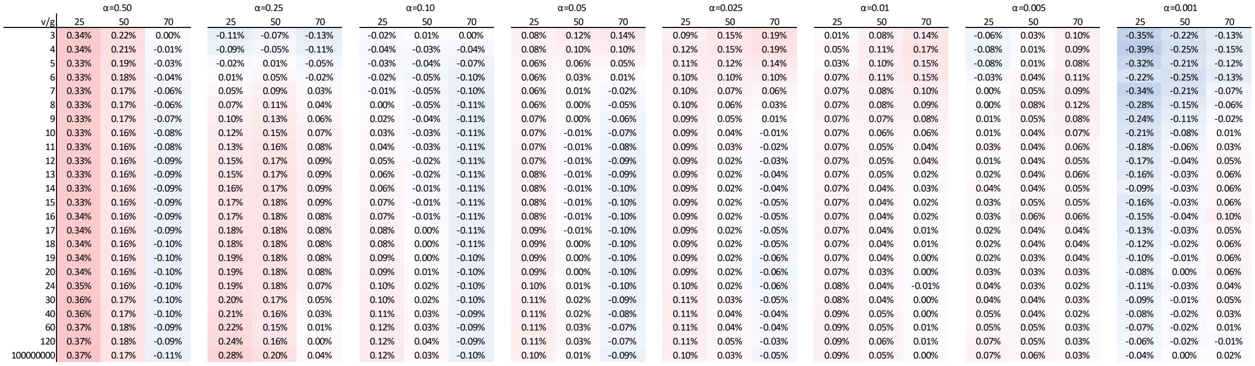
v/g	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	30	40	60	80	100
3	0.54%	0.42%	0.29%	0.18%	0.10%	0.05%	0.00%	-0.03%	-0.05%	-0.04%	-0.06%	-0.08%	-0.09%	-0.10%	-0.11%	-0.11%	-0.11%	-0.10%	-0.06%	-0.01%	0.03%	0.08%	0.12%
4	0.42%	0.34%	0.25%	0.18%	0.10%	0.06%	0.02%	-0.01%	-0.03%	-0.05%	-0.06%	-0.07%	-0.08%	-0.09%	-0.09%	-0.09%	-0.10%	-0.10%	-0.08%	-0.05%	0.02%	0.08%	0.13%
5	0.33%	0.28%	0.20%	0.14%	0.09%	0.05%	0.02%	0.00%	-0.02%	-0.04%	-0.05%	-0.06%	-0.07%	-0.07%	-0.08%	-0.08%	-0.08%	-0.08%	-0.07%	-0.04%	0.01%	0.07%	0.11%
6	0.25%	0.20%	0.14%	0.10%	0.06%	0.03%	0.01%	0.01%	-0.01%	-0.02%	-0.03%	-0.04%	-0.05%	-0.05%	-0.06%	-0.06%	-0.06%	-0.06%	-0.06%	-0.04%	0.01%	0.05%	0.08%
7	0.18%	0.17%	0.12%	0.08%	0.05%	0.03%	0.01%	0.00%	-0.01%	-0.02%	-0.03%	-0.03%	-0.04%	-0.04%	-0.05%	-0.05%	-0.05%	-0.05%	-0.04%	-0.03%	0.01%	0.04%	0.07%
8	0.15%	0.12%	0.09%	0.07%	0.05%	0.02%	0.01%	0.00%	-0.01%	-0.02%	-0.02%	-0.03%	-0.03%	-0.03%	-0.04%	-0.04%	-0.04%	-0.04%	-0.03%	-0.02%	0.01%	0.03%	0.05%
9	0.10%	0.09%	0.07%	0.05%	0.03%	0.01%	0.00%	0.00%	0.00%	0.00%	-0.02%	-0.02%	-0.03%	-0.03%	-0.03%	-0.03%	-0.03%	-0.03%	-0.03%	-0.01%	0.01%	0.02%	0.04%
10	0.08%	0.07%	0.05%	0.03%	0.03%	0.02%	0.01%	0.00%	-0.01%	-0.01%	-0.02%	-0.02%	-0.02%	-0.02%	-0.02%	-0.02%	-0.01%	-0.01%	-0.02%	-0.01%	0.01%	0.02%	0.03%
11	0.06%	0.05%	0.03%	0.02%	0.01%	0.00%	0.00%	-0.01%	0.00%	0.00%	-0.01%	-0.01%	-0.01%	-0.01%	-0.01%	-0.01%	-0.02%	-0.02%	-0.01%	-0.01%	0.01%	0.01%	0.02%
12	0.04%	0.03%	0.02%	0.01%	0.01%	0.00%	0.00%	-0.01%	-0.01%	-0.01%	-0.01%	-0.01%	-0.01%	-0.01%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.01%	0.01%	0.00%
13	0.03%	0.02%	0.01%	0.01%	0.00%	0.00%	0.00%	0.00%	-0.01%	-0.01%	-0.01%	-0.01%	-0.01%	-0.01%	-0.01%	-0.01%	-0.01%	-0.01%	0.01%	0.01%	0.01%	0.00%	0.00%
14	0.02%	0.01%	0.01%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.01%	0.00%	-0.01%
15	0.01%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.01%	0.01%	0.00%	0.00%	0.00%
16	0.00%	0.00%	-0.01%	-0.01%	-0.01%	-0.01%	-0.01%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.01%	0.01%	0.01%	0.01%	0.01%	0.00%	-0.02%
17	-0.01%	-0.01%	-0.01%	-0.01%	-0.01%	-0.01%	-0.01%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.01%	0.01%	0.01%	0.01%	0.01%	0.01%	0.01%	-0.01%	-0.02%
18	-0.01%	-0.02%	-0.02%	-0.01%	-0.01%	-0.01%	-0.01%	-0.01%	0.00%	0.00%	0.00%	0.00%	0.00%	0.01%	0.01%	0.01%	0.01%	0.01%	0.02%	0.02%	0.01%	-0.01%	-0.02%
19	-0.02%	-0.02%	-0.02%	-0.02%	-0.01%	-0.01%	-0.01%	-0.01%	0.00%	0.00%	0.00%	0.00%	0.01%	0.01%	0.01%	0.01%	0.01%	0.01%	0.02%	0.02%	0.01%	-0.01%	-0.03%
20	-0.02%	-0.03%	-0.02%	-0.02%	-0.02%	-0.01%	-0.01%	-0.01%	0.00%	0.00%	0.00%	0.00%	0.01%	0.01%	0.01%	0.01%	0.01%	0.01%	0.02%	0.02%	0.01%	-0.01%	-0.03%
24	-0.04%	-0.04%	-0.03%	-0.03%	-0.02%	-0.02%	-0.01%	-0.01%	0.00%	0.00%	0.01%	0.01%	0.01%	0.01%	0.02%	0.02%	0.02%	0.02%	0.03%	0.02%	0.01%	-0.01%	-0.04%
30	-0.05%	-0.05%	-0.05%	-0.04%	-0.03%	-0.02%	-0.01%	-0.01%	0.00%	0.00%	0.01%	0.01%	0.01%	0.02%	0.02%	0.02%	0.02%	0.03%	0.03%	0.03%	0.01%	-0.02%	-0.05%
40	-0.06%	-0.06%	-0.06%	-0.04%	-0.03%	-0.03%	-0.02%	-0.01%	0.00%	0.00%	0.01%	0.01%	0.01%	0.02%	0.03%	0.03%	0.03%	0.04%	0.04%	0.03%	0.01%	-0.02%	-0.06%
60	-0.08%	-0.08%	-0.07%	-0.05%	-0.04%	-0.03%	-0.02%	-0.01%	0.00%	0.01%	0.01%	0.02%	0.02%	0.03%	0.03%	0.03%	0.03%	0.04%	0.04%	0.04%	0.01%	-0.03%	-0.06%
120	-0.09%	-0.10%	-0.08%	-0.06%	-0.04%	-0.03%	-0.02%	-0.01%	0.00%	0.01%	0.01%	0.02%	0.02%	0.03%	0.03%	0.04%	0.04%	0.04%	0.05%	0.04%	0.01%	-0.03%	-0.07%
100000000	-0.10%	-0.11%	-0.09%	-0.06%	-0.05%	-0.03%	-0.02%	-0.01%	0.00%	0.01%	0.02%	0.02%	0.03%	0.03%	0.04%	0.04%	0.04%	0.05%	0.05%	0.04%	0.01%	-0.04%	-0.08%

v/g	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	30	40	60	80	100
3	0.45%	0.33%	0.22%	0.13%	0.07%	0.03%	-0.01%	-0.03%	-0.04%	-0.05%	-0.06%	-0.07%	-0.07%	-0.07%	-0.07%	-0.07%	-0.07%	-0.07%	-0.04%	0.00%	0.04%	0.06%	0.07%
4	0.30%	0.24%	0.16%	0.10%	0.05%	0.02%	0.00%	-0.02%	-0.03%	-0.04%	-0.04%	-0.05%	-0.05%	-0.05%	-0.05%	-0.05%	-0.05%	-0.05%	-0.03%	-0.01%	0.03%	0.05%	0.06%
5	0.20%	0.16%	0.11%	0.07%	0.04%	0.01%	0.00%	-0.01%	-0.02%	-0.03%	-0.03%	-0.03%	-0.04%	-0.04%	-0.04%	-0.04%	-0.04%	-0.04%	-0.02%	0.00%	0.02%	0.03%	0.03%
6	0.12%	0.10%	0.06%	0.04%	0.02%	0.00%	0.00%	-0.01%	-0.02%	-0.02%	-0.02%	-0.02%	-0.02%	-0.02%	-0.02%	-0.02%	-0.02%	-0.02%	0.00%	0.01%	0.02%	0.02%	0.01%
7	0.07%	0.05%	0.03%	0.01%	0.00%	0.00%	-0.01%	-0.01%	-0.01%	-0.01%	-0.01%	-0.01%	-0.01%	-0.01%	-0.01%	-0.01%	-0.01%	0.00%	0.01%	0.02%	0.02%	0.01%	-0.01%
8	0.02%	0.01%	0.00%	-0.01%	-0.01%	-0.01%	-0.01%	-0.01%	-0.01%	-0.01%	0.00%	0.00%	0.00%	0.00%	0.00%	0.01%	0.01%	0.01%	0.02%	0.02%	0.01%	0.00%	-0.03%
9	-0.01%	-0.02%	-0.03%	-0.03%	-0.02%	-0.02%	-0.02%	-0.01%	-0.01%	0.00%	0.00%	0.00%	0.01%	0.01%	0.01%	0.01%	0.02%	0.02%	0.03%	0.03%	0.01%	-0.01%	-0.04%
10	-0.04%	-0.05%	-0.04%	-0.04%	-0.03%	-0.03%	-0.02%	-0.01%	-0.01%	0.00%	0.00%	0.01%	0.01%	0.02%	0.02%	0.02%	0.02%	0.03%	0.04%	0.04%	0.02%	-0.02%	-0.05%
11	-0.06%	-0.07%	-0.06%	-0.05%	-0.04%	-0.03%	-0.02%	-0.01%	-0.01%	0.00%	0.01%	0.01%	0.02%	0.02%	0.03%	0.03%	0.03%	0.03%	0.05%	0.04%	0.01%	-0.02%	-0.06%
12	-0.08%	-0.08%	-0.07%	-0.06%	-0.05%	-0.03%	-0.02%	-0.01%	0.00%	0.00%	0.01%	0.02%	0.02%	0.03%	0.03%	0.03%	0.04%	0.04%	0.05%	0.05%	0.01%	-0.03%	-0.07%
13	-0.09%	-0.10%	-0.08%	-0.07%	-0.05%	-0.04%	-0.02%	-0.01%	0.00%	0.00%	0.01%	0.02%	0.02%	0.03%	0.03%	0.04%	0.04%	0.04%	0.06%	0.05%	0.01%	-0.03%	-0.08%
14	-0.11%	-0.11%	-0.09%	-0.07%	-0.06%	-0.04%	-0.03%	-0.01%	0.00%	0.01%	0.01%	0.02%	0.03%	0.03%	0.04%	0.04%	0.05%	0.05%	0.06%	0.05%	0.01%	-0.04%	-0.09%
15	-0.12%	-0.12%	-0.10%	-0.08%	-0.06%	-0.04%	-0.03%	-0.01%	0.00%	0.01%	0.02%	0.02%	0.03%	0.04%	0.04%	0.05%	0.05%	0.05%	0.06%	0.06%	0.01%	-0.04%	-0.09%
16	-0.13%	-0.13%	-0.11%	-0.08%	-0.06%	-0.05%	-0.03%	-0.01%	0.00%	0.01%	0.02%	0.03%	0.03%	0.04%	0.04%	0.05%	0.05%	0.06%	0.07%	0.06%	0.01%	-0.04%	-0.10%
17	-0.13%	-0.13%	-0.11%	-0.09%	-0.07%	-0.05%	-0.03%	-0.02%	0.00%	0.01%	0.02%	0.03%	0.03%	0.04%	0.05%	0.05%	0.05%	0.06%	0.07%	0.06%	0.01%	-0.04%	-0.10%
18	-0.14%	-0.14%	-0.12%	-0.09%	-0.07%	-0.05%	-0.03%	-0.02%	0.00%	0.01%	0.02%	0.03%	0.04%	0.04%	0.05%	0.05%	0.06%	0.06%	0.07%	0.06%	0.01%	-0.04%	-0.11%
19	-0.15%	-0.14%	-0.12%	-0.10%	-0.07%	-0.05%	-0.03%	-0.02%	0.00%	0.01%	0.02%	0.03%	0.04%	0.04%	0.05%	0.05%	0.06%	0.06%	0.07%	0.06%	0.01%	-0.05%	-0.11%
20	-0.15%	-0.15%	-0.12%	-0.10%	-0.07%	-0.05%	-0.03%	-0.01%	0.00%	0.01%	0.02%	0.03%	0.04%	0.04%	0.05%	0.06%	0.06%	0.06%	0.08%	0.06%	0.01%	-0.05%	-0.11%
24	-0.17%	-0.16%	-0.14%	-0.11%	-0.08%	-0.06%	-0.03%	-0.02%	0.00%	0.01%	0.02%	0.03%	0.04%	0.05%	0.06%	0.06%	0.07%	0.07%	0.08%	0.07%	0.02%	-0.05%	-0.12%
30	-0.18%	-0.18%	-0.15%	-0.12%	-0.09%	-0.06%	-0.04%	-0.02%	0.00%	0.01%	0.03%	0.04%	0.05%	0.05%	0.06%	0.07%	0.07%	0.08%	0.09%	0.07%	0.01%	-0.06%	-0.13%
40	-0.20%	-0.19%	-0.16%	-0.12%	-0.09%	-0.06%	-0.04%	-0.02%	0.00%	0.02%	0.03%	0.04%	0.05%	0.06%	0.07%	0.07%	0.08%	0.08%	0.09%	0.08%	0.02%	-0.06%	-0.14%
60	-0.21%	-0.21%	-0.17%	-0.13%	-0.10%	-0.07%	-0.04%	-0.02%	0.00%	0.02%	0.03%	0.04%	0.05%	0.06%	0.07%	0.08%	0.08%	0.09%	0.10%	0.08%	0.02%	-0.07%	-0.15%
120	-0.23%	-0.22%	-0.18%	-0.14%	-0.10%	-0.07%	-0.04%	-0.02%	0.00%	0.02%	0.03%	0.05%	0.06%	0.07%	0.07%	0.08%	0.09%	0.09%	0.11%	0.09%	0.02%	-0.07%	-0.16%
100000000	-0.24%	-0.23%	-0.19%	-0.15%	-0.11%	-0.07%	-0.04%	-0.02%	0.00%	0.02%	0.04%	0.05%	0.06%	0.07%	0.08%	0.09%	0.09%	0.10%	0.11%	0.09%	0.02%	-0.07%	-0.17%

Figure 3. Error map of α = 0.01 (top) and α = 0.05 (bottom). Additional error maps for other α values can be found in Figures S2-S7. A positive error means that the approximated value is higher than the tabulated value [i.e., (estimated–exact)/exact)].

5.1. Interpolation and Extrapolation in g, (Number of Groups)

The quantiles (t_D values) were calculated for values of g, not specifically used when developing the correlation in Equation 6. But as g is incorporated into Equation 6, it can be directly implemented. In Figure 4, the error map between calculated t_D values based on Equation 6 and those obtained by SAS Studio is shown. As is noted, the errors were below ±0.4%. Extrapolation to g > 100 is not recommended, as it creates larger errors (data not shown).



	α=0.50			α=0.25			α=0.10			α=0.05			α=0.025			α=0.01			α=0.005			α=0.001		
v/g	25	50	70	25	50	70	25	50	70	25	50	70	25	50	70	25	50	70	25	50	70	25	50	70
3	0.34%	0.22%	0.00%	-0.11%	-0.07%	-0.13%	-0.02%	0.01%	0.00%	0.08%	0.12%	0.14%	0.09%	0.15%	0.19%	0.01%	0.08%	0.14%	-0.06%	0.03%	0.10%	-0.35%	-0.22%	-0.13%
4	0.34%	0.21%	-0.01%	-0.09%	-0.05%	-0.11%	-0.04%	-0.03%	-0.04%	0.08%	0.10%	0.10%	0.12%	0.15%	0.19%	0.05%	0.11%	0.17%	-0.08%	0.01%	0.09%	-0.39%	-0.25%	-0.15%
5	0.33%	0.19%	-0.03%	-0.02%	0.01%	-0.05%	-0.03%	-0.04%	-0.07%	0.06%	0.06%	0.05%	0.11%	0.12%	0.14%	0.03%	0.10%	0.15%	-0.08%	0.01%	0.08%	-0.32%	-0.21%	-0.12%
6	0.33%	0.18%	-0.04%	0.01%	0.05%	-0.02%	-0.02%	-0.05%	-0.10%	0.06%	0.03%	0.01%	0.10%	0.10%	0.10%	0.07%	0.11%	0.15%	-0.03%	0.04%	0.11%	-0.22%	-0.25%	-0.13%
7	0.33%	0.17%	-0.06%	0.05%	0.09%	0.03%	-0.01%	-0.05%	-0.10%	0.06%	0.01%	-0.02%	0.10%	0.07%	0.06%	0.07%	0.08%	0.10%	0.00%	0.05%	0.09%	-0.34%	-0.21%	-0.07%
8	0.33%	0.17%	-0.06%	0.07%	0.11%	0.04%	0.00%	-0.05%	-0.11%	0.06%	0.00%	-0.05%	0.10%	0.06%	0.03%	0.07%	0.08%	0.09%	0.00%	0.08%	0.12%	-0.28%	-0.15%	-0.06%
9	0.33%	0.17%	-0.07%	0.10%	0.13%	0.06%	0.02%	-0.04%	-0.11%	0.07%	0.00%	-0.06%	0.09%	0.05%	0.01%	0.07%	0.07%	0.08%	0.01%	0.05%	0.08%	-0.24%	-0.11%	-0.02%
10	0.33%	0.16%	-0.08%	0.12%	0.15%	0.07%	0.03%	-0.03%	-0.11%	0.07%	-0.01%	-0.07%	0.09%	0.04%	-0.01%	0.07%	0.06%	0.06%	0.01%	0.04%	0.07%	-0.21%	-0.08%	0.01%
11	0.33%	0.16%	-0.08%	0.13%	0.16%	0.08%	0.04%	-0.03%	-0.11%	0.07%	-0.01%	-0.08%	0.09%	0.03%	-0.02%	0.07%	0.05%	0.04%	0.03%	0.04%	0.06%	-0.18%	-0.06%	0.03%
12	0.33%	0.16%	-0.09%	0.15%	0.17%	0.09%	0.05%	-0.02%	-0.11%	0.07%	-0.01%	-0.09%	0.09%	0.02%	-0.03%	0.07%	0.05%	0.04%	0.01%	0.04%	0.05%	-0.17%	-0.04%	0.05%
13	0.33%	0.16%	-0.09%	0.15%	0.17%	0.09%	0.06%	-0.02%	-0.11%	0.08%	-0.01%	-0.09%	0.09%	0.02%	-0.04%	0.07%	0.05%	0.02%	0.02%	0.04%	0.06%	-0.16%	-0.03%	0.06%
14	0.33%	0.16%	-0.09%	0.16%	0.17%	0.09%	0.06%	-0.01%	-0.11%	0.08%	-0.01%	-0.10%	0.09%	0.02%	-0.04%	0.07%	0.04%	0.03%	0.04%	0.04%	0.05%	-0.09%	-0.03%	0.06%
15	0.33%	0.16%	-0.09%	0.17%	0.18%	0.09%	0.07%	-0.01%	-0.11%	0.08%	-0.01%	-0.10%	0.09%	0.02%	-0.05%	0.07%	0.04%	0.02%	0.03%	0.05%	0.05%	-0.16%	-0.03%	0.06%
16	0.34%	0.16%	-0.09%	0.17%	0.18%	0.08%	0.07%	-0.01%	-0.11%	0.08%	-0.01%	-0.10%	0.09%	0.02%	-0.05%	0.07%	0.04%	0.02%	0.03%	0.06%	0.06%	-0.15%	-0.04%	0.10%
17	0.34%	0.16%	-0.09%	0.18%	0.18%	0.08%	0.08%	0.00%	-0.11%	0.09%	-0.01%	-0.10%	0.09%	0.02%	-0.05%	0.07%	0.04%	0.01%	0.02%	0.04%	0.04%	-0.13%	-0.03%	0.05%
18	0.34%	0.16%	-0.10%	0.18%	0.18%	0.08%	0.08%	0.00%	-0.11%	0.09%	0.00%	-0.10%	0.09%	0.02%	-0.05%	0.07%	0.04%	0.01%	0.02%	0.04%	0.04%	-0.12%	-0.02%	0.06%
19	0.34%	0.16%	-0.10%	0.19%	0.18%	0.08%	0.09%	0.00%	-0.10%	0.09%	0.00%	-0.10%	0.09%	0.02%	-0.06%	0.07%	0.04%	0.00%	0.02%	0.03%	0.04%	-0.10%	-0.01%	0.06%
20	0.34%	0.16%	-0.10%	0.19%	0.18%	0.08%	0.09%	0.01%	-0.10%	0.09%	0.00%	-0.10%	0.09%	0.02%	-0.06%	0.07%	0.03%	0.00%	0.03%	0.03%	0.03%	-0.08%	0.00%	0.06%
24	0.35%	0.16%	-0.10%	0.19%	0.18%	0.07%	0.10%	0.02%	-0.10%	0.10%	0.01%	-0.10%	0.10%	0.02%	-0.06%	0.08%	0.04%	-0.01%	0.04%	0.03%	0.02%	-0.11%	-0.03%	0.04%
30	0.36%	0.17%	-0.10%	0.20%	0.17%	0.05%	0.10%	0.02%	-0.10%	0.11%	0.02%	-0.09%	0.11%	0.03%	-0.05%	0.08%	0.04%	0.00%	0.04%	0.04%	0.03%	-0.09%	-0.01%	0.05%
40	0.36%	0.17%	-0.10%	0.21%	0.16%	0.03%	0.11%	0.03%	-0.09%	0.11%	0.02%	-0.08%	0.11%	0.04%	-0.04%	0.09%	0.05%	0.00%	0.05%	0.04%	0.02%	-0.08%	-0.02%	0.03%
60	0.37%	0.18%	-0.09%	0.22%	0.15%	0.01%	0.12%	0.03%	-0.09%	0.11%	0.03%	-0.07%	0.11%	0.04%	-0.04%	0.09%	0.05%	0.01%	0.05%	0.05%	0.03%	-0.07%	-0.02%	0.01%
120	0.37%	0.18%	-0.09%	0.24%	0.16%	0.00%	0.12%	0.04%	-0.09%	0.11%	0.03%	-0.07%	0.11%	0.05%	-0.03%	0.09%	0.06%	0.01%	0.07%	0.05%	0.03%	-0.06%	-0.02%	-0.01%
100000000	0.37%	0.17%	-0.11%	0.28%	0.20%	0.04%	0.12%	0.03%	-0.10%	0.10%	0.01%	-0.09%	0.10%	0.03%	-0.05%	0.09%	0.05%	0.00%	0.07%	0.06%	0.03%	-0.04%	0.00%	0.02%

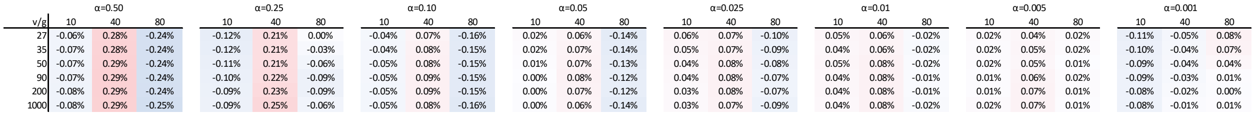
Figure 4. Error map of interpolation for g = 25, 50, and 70 with several α values and df (i.e., v) values.

Table 1. Maximum and average errors in determining t_D with a non-iterative method, using the user-defined TDINV function in Excel.

α	**Max positive error**	**Max negative error**	**Mean absolute error**	**Comments where maximum errors are noted**
0.50	0.37%	-0.87%	0.27%	With low g
0.25	0.27%	-0.70%	0.17%	With low g with high v
0.10	0.34%	-0.37%	0.08%	
0.050	0.45%	-0.24%	0.05%	
0.025	0.50%	-0.17%	0.04%	
0.010	0.54%	-0.11%	0.03%	With low g with low v
0.005	0.53%	-0.11%	0.05%	With low g with low v
0.001	0.60%	-0.14%	0.04%	With low g and low v

5.2. Interpolation in df (i.e., v)

Values of t_D were calculated via the interpolation method shown in Step 1-3 (Equations 7-9) for df (i.e., v) values not available when developing the correlations for selected group numbers (g = 10, 40, and 80) and for several α values. The error map is shown in Figure 5. As is noted, the errors were below ±0.3% with mean absolute error of 0.07%.

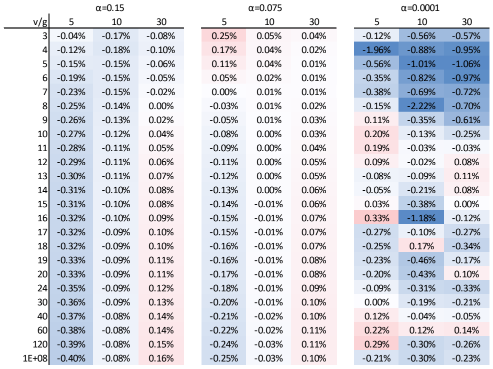


	α=0.50			α=0.25			α=0.10			α=0.05			α=0.025			α=0.01			α=0.005			α=0.001		
v/g	10	40	80	10	40	80	10	40	80	10	40	80	10	40	80	10	40	80	10	40	80	10	40	80
27	-0.06%	0.28%	-0.24%	-0.12%	0.21%	0.00%	-0.04%	0.07%	-0.16%	0.02%	0.06%	-0.14%	0.06%	0.07%	-0.10%	0.05%	0.06%	-0.02%	0.02%	0.04%	0.02%	-0.11%	-0.05%	0.08%
35	-0.07%	0.28%	-0.24%	-0.12%	0.21%	-0.03%	-0.04%	0.08%	-0.15%	0.02%	0.07%	-0.14%	0.05%	0.07%	-0.09%	0.04%	0.06%	-0.02%	0.02%	0.05%	0.02%	-0.10%	-0.04%	0.07%
50	-0.07%	0.29%	-0.24%	-0.11%	0.21%	-0.06%	-0.05%	0.08%	-0.15%	0.01%	0.07%	-0.13%	0.04%	0.08%	-0.08%	0.05%	0.08%	-0.02%	0.02%	0.05%	0.01%	-0.09%	-0.04%	0.04%
90	-0.07%	0.29%	-0.24%	-0.10%	0.22%	-0.09%	-0.05%	0.09%	-0.15%	0.00%	0.08%	-0.12%	0.04%	0.08%	-0.07%	0.04%	0.08%	-0.01%	0.01%	0.06%	0.02%	-0.09%	-0.03%	0.01%
200	-0.08%	0.29%	-0.24%	-0.09%	0.23%	-0.09%	-0.05%	0.09%	-0.15%	0.00%	0.07%	-0.12%	0.03%	0.08%	-0.07%	0.04%	0.08%	-0.01%	0.01%	0.07%	0.01%	-0.08%	-0.02%	0.00%
1000	-0.08%	0.29%	-0.25%	-0.09%	0.25%	-0.06%	-0.05%	0.08%	-0.16%	0.00%	0.06%	-0.14%	0.03%	0.07%	-0.09%	0.04%	0.08%	-0.02%	0.02%	0.07%	0.01%	-0.08%	-0.01%	0.01%

Figure 5. Error map of interpolation for g = 10, 40, and 80 with several α values and interpolated df (i.e., v) values.

5.3. Interpolation and Extrapolation in α

The method was developed using a range of α values from 0.001 to 0.5, and the most common used values of 0.01 and 0.05 where included. However, as the method also allow for interpolation, we calculated the error between t_D values based on Steps 4-6 (Equation 10-12, and Equation 9) and those obtained by SAS Studio. The error map is shown in Figure 6. Also included, is the error when α is extrapolated to 0.0001. As noted, the interpolation works quite well with errors less than ±0.4%, but the extrapolation contains several errors greater than 1%. However, it should be noted that the majority of these occurred (for the limited cases studied) when df (i.e., v) < g. This is rare situation, and only occurs when there are fewer than two observations in multiple groups. The mean absolute error for the interpolation and extrapolation cases was 0.16%.



	α=0.15			α=0.075			α=0.0001		
v/g	5	10	30	5	10	30	5	10	30
3	-0.04%	-0.17%	-0.08%	0.25%	0.05%	0.04%	-0.12%	-0.56%	-0.57%
4	-0.12%	-0.18%	-0.10%	0.17%	0.04%	0.02%	-1.96%	-0.88%	-0.95%
5	-0.15%	-0.15%	-0.06%	0.11%	0.04%	0.01%	-0.56%	-1.01%	-1.06%
6	-0.19%	-0.15%	-0.05%	0.05%	0.02%	0.01%	-0.35%	-0.82%	-0.97%
7	-0.23%	-0.15%	-0.02%	0.00%	0.01%	0.01%	-0.38%	-0.69%	-0.72%
8	-0.25%	-0.14%	0.00%	-0.03%	0.01%	0.02%	-0.15%	-2.22%	-0.70%
9	-0.26%	-0.13%	0.02%	-0.05%	0.01%	0.03%	0.11%	-0.35%	-0.61%
10	-0.27%	-0.12%	0.04%	-0.08%	0.00%	0.03%	0.20%	-0.13%	-0.25%
11	-0.28%	-0.11%	0.05%	-0.09%	0.00%	0.04%	0.19%	-0.03%	-0.03%
12	-0.29%	-0.11%	0.06%	-0.11%	0.00%	0.05%	0.09%	-0.02%	0.08%
13	-0.30%	-0.11%	0.07%	-0.12%	0.00%	0.05%	-0.08%	-0.09%	0.11%
14	-0.31%	-0.10%	0.08%	-0.13%	0.00%	0.06%	-0.05%	-0.21%	0.08%
15	-0.31%	-0.10%	0.08%	-0.14%	-0.01%	0.06%	0.03%	-0.38%	0.00%
16	-0.32%	-0.10%	0.09%	-0.15%	-0.01%	0.07%	0.33%	-1.18%	-0.12%
17	-0.32%	-0.09%	0.10%	-0.15%	-0.01%	0.07%	-0.27%	-0.10%	-0.27%
18	-0.32%	-0.09%	0.10%	-0.16%	-0.01%	0.07%	-0.25%	0.17%	-0.34%
19	-0.33%	-0.09%	0.11%	-0.16%	-0.01%	0.08%	-0.23%	-0.46%	-0.17%
20	-0.33%	-0.09%	0.11%	-0.17%	-0.01%	0.08%	-0.20%	-0.43%	0.10%
24	-0.35%	-0.09%	0.12%	-0.18%	-0.01%	0.09%	-0.09%	-0.31%	-0.33%
30	-0.36%	-0.09%	0.13%	-0.20%	-0.01%	0.10%	0.00%	-0.19%	-0.21%
40	-0.37%	-0.08%	0.14%	-0.21%	-0.02%	0.10%	0.12%	-0.04%	-0.05%
60	-0.38%	-0.08%	0.14%	-0.22%	-0.02%	0.11%	0.22%	0.12%	0.14%
120	-0.39%	-0.08%	0.15%	-0.24%	-0.03%	0.11%	0.29%	-0.30%	-0.26%
1E+08	-0.40%	-0.08%	0.16%	-0.25%	-0.03%	0.10%	-0.21%	-0.30%	-0.23%

Figure 6. Error map for g = 5, 10, and 30 with two interpolated and one extrapolated α values and several df (i.e., v) values.

5.4. An Example of Implementation

The comparison of means according to Dunnett using the example previously mentioned has been implemented in an Excel spreadsheet, which includes the TDINV function for t_D values and the QINVG function for Q values [6], together with evaluation of variance normality and equivalency [8, 13, 14]. In addition to the Dunnett's test, Fisher LSD, Tukey HSD, Fisher-Hayter, Scheffe, and REGWQ [3, 15] are available in the example workbook in the Supplementary Materials (Figure S8). Details of how to create such an Excel workbook are provided by Klasson [16].

5.5. Limitations

The method developed is restricted to $2 \le g \le 100$, $2 \le$ df (i.e., v) $\le 100{,}000{,}000$, and $0.001 \le \alpha \le 0.5$. In the created TDINV Excel function and the other spreadsheet implementation, a #VALUE error will be reported if any of the limits are exceeded.

6. SUPPLEMENTARY MATERIALS

The following supporting information follows the references: Table S1: Listing of A_1 and A_2 values needed in Equation 6; Code S1: Visual Basic code for TDINV(α, g, v); Figure S1: Excel spreadsheet with method suitable for Google Sheet implementation; Figure S2: Error map of α = 0.50, comparing the developed method with values from SAS Studio; Figure S3: Error map of α = 0.25; Figure S4: Error map of α = 0.10; Figure S5: Error map of α = 0.025; Figure S6: Error map of α = 0.005; Figure S7: Error map of α = 0.001; Figure S8: Example Excel workbook of Dunnett's (and other) test(s).

NOMENCLATURE

A_1 and A_2 = polynomial parameters in Equation 6

α = confidence level, normally 0.05

b_1 and b_2 and b_2 = polynomial parameters in Equation 11

CV = critical value

df_w and df and v = degrees of freedom within the groups (e.g., from an ANOVA table)

g = number of groups

MS_w = mean sum of squares within the groups (from an ANOVA table)

n_c = number of data points in control group

n_t = number of data points in treatment group

Q = Q values from the Studentized range distribution

SF = safety factor

t_d = t values from the Dunnett's t_d distribution

t = t statistics, Equation 1

x = function of the standard normal distribution, Equation 12

$\overline{x}_t$ = mean of treatment group

$\overline{x}_c$ = mean of control group

$y = t_D(\alpha,g,v) / t(\alpha,v)$

$z = \ln(y + g/v)$

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Mention of trade names or commercial products in this publication is solely for the purpose of providing specific information and does not imply recommendation or endorsement by the U.S. Department of Agriculture. USDA is an equal opportunity provider and employer.

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Supporting Materials

A Non-Iterative Approximation for Critical Values for the Dunnett's Test

K. Thomas Klasson

Table S1. Listing of A_1 and A_2 values needed in Equation 6.

α >>	0.5		0.25		0.1		0.05		0.025		0.01		0.005		0.001	
v (df)	A1	A2	A1	A2	A1	A2	A1	A2	A1	A2	A1	A2	A1	A2	A1	A2
000000003	0.72397914	-0.04288019	0.46150139	-0.02255796	0.35861359	-0.01452118	0.32616335	-0.01200206	0.30841731	-0.01062847	0.29574527	-0.00964923	0.29016302	-0.00920698	0.28472278	-0.00879441
000000004	0.72302968	-0.04267121	0.44256976	-0.02158658	0.32808423	-0.01293121	0.28964214	-0.01005283	0.26726281	-0.00839380	0.24995902	-0.00712259	0.24200611	-0.00653938	0.23180479	-0.00582871
000000005	0.72214379	-0.04250446	0.43086420	-0.02109903	0.30961672	-0.01215148	0.26771800	-0.00910992	0.24255558	-0.00731645	0.22216728	-0.00585761	0.21232058	-0.00516829	0.19718640	-0.00389867
000000006	0.72133342	-0.04237675	0.42289904	-0.02083655	0.29734810	-0.01174618	0.25324552	-0.00861711	0.22626923	-0.00673667	0.20387187	-0.00518973	0.19276888	-0.00442809	0.17634825	-0.00320779
000000007	0.72061179	-0.04228144	0.41714139	-0.02069517	0.28865489	-0.01153042	0.24305997	-0.00835105	0.21485945	-0.00642300	0.19122857	-0.00484190	0.17899201	-0.00400126	0.16223699	-0.00306789
000000008	0.71997833	-0.04221184	0.41278848	-0.02062177	0.28220004	-0.01141764	0.23553716	-0.00820633	0.20647000	-0.00624937	0.18185070	-0.00462757	0.16923750	-0.00383582	0.15051453	-0.00260942
000000009	0.71942349	-0.04216181	0.40938449	-0.02058756	0.27722674	-0.01136234	0.22978761	-0.00813343	0.20006855	-0.00615437	0.17471505	-0.00450514	0.16154368	-0.00365636	0.14186299	-0.00242517
000000010	0.71893973	-0.04212728	0.40665749	-0.02057910	0.27328175	-0.01133920	0.22524749	-0.00809856	0.19503764	-0.00610563	0.16911544	-0.00443514	0.15556546	-0.00357240	0.13508135	-0.00230833
000000011	0.71851921	-0.04210515	0.40442553	-0.02058619	0.27008704	-0.01133772	0.22157442	-0.00808460	0.19099774	-0.00608690	0.16462510	-0.00439832	0.15079055	-0.00352775	0.12976971	-0.00226627
000000012	0.71814945	-0.04209195	0.40256735	-0.02060306	0.26744841	-0.01134847	0.21855312	-0.00808657	0.18766481	-0.00607987	0.16095307	-0.00438474	0.14687493	-0.00350378	0.12548869	-0.00225737
000000013	0.71782583	-0.04208613	0.40099769	-0.02062592	0.26523424	-0.01136641	0.21602394	-0.00809698	0.18486845	-0.00607865	0.15789853	-0.00438259	0.14362714	-0.00350203	0.12181418	-0.00222807
000000014	0.71753998	-0.04208571	0.39965537	-0.02065239	0.26335088	-0.01138844	0.21387686	-0.00811227	0.18250553	-0.00608695	0.15530919	-0.00438782	0.14089654	-0.00350181	0.11877835	-0.00221835
000000015	0.71728839	-0.04208987	0.39849501	-0.02068080	0.26173015	-0.01141257	0.21203224	-0.00813028	0.18047769	-0.00609872	0.15307232	-0.00439072	0.13856170	-0.00351120	0.11601319	-0.00217031
000000016	0.71706417	-0.04209715	0.39748253	-0.02071012	0.26031962	-0.01143714	0.21043090	-0.00814963	0.17871892	-0.00611244	0.15111669	-0.00439052	0.13653337	-0.00352185	0.11348401	-0.00209587
000000017	0.71686413	-0.04210694	0.39659184	-0.02073963	0.25907851	-0.01146078	0.20902793	-0.00816941	0.17717916	-0.00612709	0.14942257	-0.00439920	0.13472445	-0.00352114	0.11143492	-0.00209112
000000018	0.71668529	-0.04211857	0.39580245	-0.02076887	0.25797963	-0.01148379	0.20778870	-0.00818907	0.17582027	-0.00614210	0.14792849	-0.00440883	0.13311054	-0.00351606	0.10950163	-0.00204780
000000019	0.71652541	-0.04213181	0.39509832	-0.02079750	0.25700056	-0.01150615	0.20668084	-0.00820680	0.17461200	-0.00615698	0.14660126	-0.00441895	0.13170389	-0.00352320	0.10803667	-0.00208734
000000020	0.71638132	-0.04214617	0.39446630	-0.02082529	0.25611886	-0.01152693	0.20567813	-0.00822131	0.17353095	-0.00617157	0.14541427	-0.00442918	0.13044657	-0.00353080	0.10684425	-0.00215181
000000024	0.71592629	-0.04220851	0.39247792	-0.02092678	0.25337841	-0.01160920	0.20254960	-0.00828078	0.17014647	-0.00622462	0.14170402	-0.00446851	0.12652064	-0.00356208	0.10297929	-0.00228433
000000030	0.71548581	-0.04230354	0.39050886	-0.02104877	0.25067306	-0.01170275	0.19949985	-0.00835910	0.16675491	-0.00627614	0.13806620	-0.00451714	0.12267891	-0.00360340	0.09854307	-0.00223936
000000040	0.71507852	-0.04244185	0.38856087	-0.02119161	0.24800445	-0.01180770	0.19648874	-0.00844366	0.16346955	-0.00634590	0.13451202	-0.00458544	0.11891984	-0.00365330	0.09441685	-0.00228192
000000060	0.71472243	-0.04263446	0.38663389	-0.02135459	0.24537091	-0.01192204	0.19352367	-0.00853648	0.16024078	-0.00642410	0.13091916	-0.00463190	0.11529640	-0.00373290	0.09040329	-0.00233387
000000120	0.71444644	-0.04289677	0.38473372	-0.02153718	0.24277101	-0.01204387	0.19060503	-0.00863703	0.15706958	-0.00651084	0.12747915	-0.00470721	0.11162556	-0.00379309	0.08649381	-0.00239241
100000000	0.71426748	-0.04323940	0.38283858	-0.02172935	0.24020454	-0.01217198	0.18773403	-0.00874542	0.15395724	-0.00660656	0.12411223	-0.00479232	0.10808061	-0.00386795	0.08267812	-0.00245567

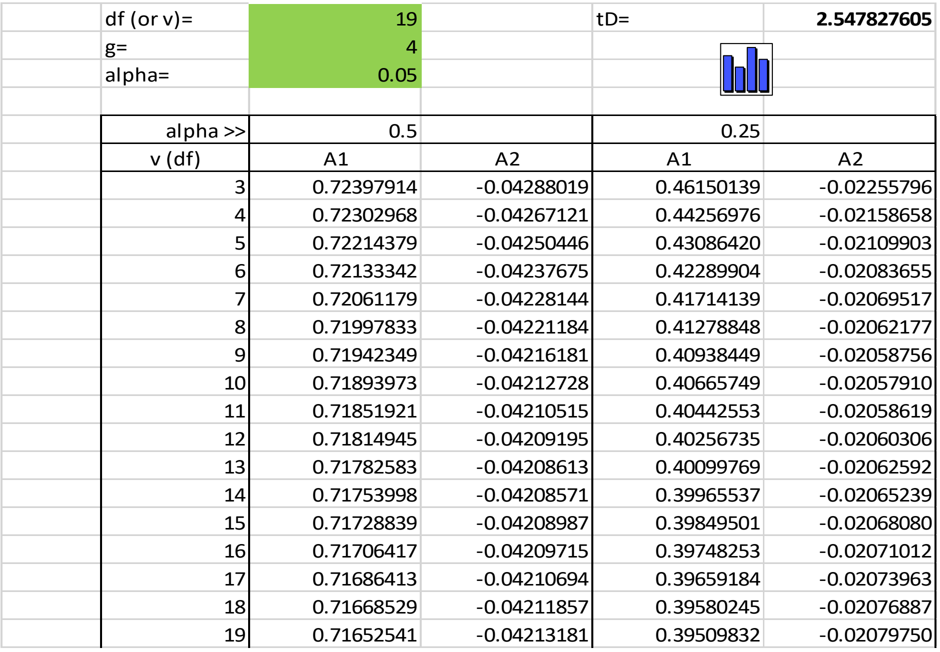
Code S1. Visual Basic code for TDINV(α, g, v)

```
Function TDINV(a, g, v)
'By K. Thomas Klasson, USDA-ARS, New Orleans, 2022
Dim i, j, iMin, iMax As Integer
Dim df(24), v_low, v_high As Long
Dim tDTable(24), tDLine(2) As String
Dim y(8), alpha(8), y2_low(8), y2_high(8), y_temp As Single
Dim z(), x(), x_temp, z_temp As Variant

'Limitations: Check if they have been exceeded.  If so, quit with error message.

If a > 0.5 Or a < 0.001 Or g < 2 Or g > 100 Or v < 2 Then
    y_temp = 1 / 0
    GoTo EndTDINV:
End If

'                      |    p=0.5, a=0.5   |   p=0.75, a=0.25  |   p=0.90, a=0.10  |   p=0.95,
a=0.05  |  p=0.975, a=0.025 |  p=0.99, a=0.01   |  p=0.995, a=0.005 | p=0.999, a=0.001
'                 DF   |    A1        A2        A1       A2         A1        A2        A1
A2       A1        A2        A1        A2        A1        A2        A1        A2
'                 1         2         3         4         5         6         7         8
9        10        11        12        13        14        15        16        17
'
1234567890123456789012345678901234567890123456789012345678901234567890123456789012345678901234567
8901234567890123456789012345678901234567890123456789012345678901234567890123456789012
tDTable(1) = "000000003  0.72397914 -0.04288019  0.46150139 -0.02255796  0.35861359 -0.01452118
0.32616335 -0.01200206  0.30841731 -0.01062847  0.29574527 -0.00964923  0.29016302 -0.00920698
0.28472278 -0.00879441"
tDTable(2) = "000000004  0.72302968 -0.04267121  0.44256976 -0.02158658  0.32808423 -0.01293121
0.28964214 -0.01005283  0.26726281 -0.00839380  0.24995902 -0.00712259  0.24200611 -0.00653938
0.23180479 -0.00582871"
tDTable(3) = "000000005  0.72214379 -0.04250446  0.43086420 -0.02109903  0.30961672 -0.01215148
0.26771800 -0.00910992  0.24255558 -0.00731645  0.22216728 -0.00585761  0.21232058 -0.00516829
0.19718640 -0.00389867"
tDTable(4) = "000000006  0.72133342 -0.04237675  0.42289904 -0.02083655  0.29734810 -0.01174618
0.25324552 -0.00861711  0.22626923 -0.00673667  0.20387187 -0.00518973  0.19276888 -0.00442809
0.17634825 -0.00320779"
tDTable(5) = "000000007  0.72061179 -0.04228144  0.41714139 -0.02069517  0.28865489 -0.01153042
0.24305997 -0.00835105  0.21485945 -0.00642300  0.19122857 -0.00484190  0.17899201 -0.00400126
0.16223699 -0.00306789"
tDTable(6) = "000000008  0.71997833 -0.04221184  0.41278848 -0.02062177  0.28220004 -0.01141764
0.23553716 -0.00820633  0.20647000 -0.00624937  0.18185070 -0.00462757  0.16923750 -0.00383582
0.15051453 -0.00260942"
tDTable(7) = "000000009  0.71942349 -0.04216181  0.40938449 -0.02058756  0.27722674 -0.01136234
0.22978761 -0.00813343  0.20006855 -0.00615437  0.17471505 -0.00450514  0.16154368 -0.00365636
0.14186299 -0.00242517"
tDTable(8) = "000000010  0.71893973 -0.04212728  0.40665749 -0.02057910  0.27328175 -0.01133920
0.22524749 -0.00809856  0.19503764 -0.00610563  0.16911544 -0.00443514  0.15556546 -0.00357240
0.13508135 -0.00230833"
tDTable(9) = "000000011  0.71851921 -0.04210515  0.40442553 -0.02058619  0.27008704 -0.01133772
0.22157442 -0.00808460  0.19099774 -0.00608690  0.16462510 -0.00439832  0.15079055 -0.00352775
0.12976971 -0.00226627"
tDTable(10) = "000000012  0.71814945 -0.04209195  0.40256735 -0.02060306  0.26744841 -0.01134847
0.21855312 -0.00808657  0.18766481 -0.00607987  0.16095307 -0.00438474  0.14687493 -0.00350378
0.12548869 -0.00225737"
tDTable(11) = "000000013  0.71782583 -0.04208613  0.40099769 -0.02062592  0.26523424 -0.01136641
0.21602394 -0.00809698  0.18486845 -0.00607865  0.15789853 -0.00438259  0.14362714 -0.00350203
0.12181418 -0.00222807"
tDTable(12) = "000000014  0.71753998 -0.04208571  0.39965537 -0.02065239  0.26335088 -0.01138844
0.21387686 -0.00811227  0.18250553 -0.00608695  0.15530919 -0.00438782  0.14089654 -0.00350181
0.11877835 -0.00221835"
tDTable(13) = "000000015  0.71728839 -0.04208987  0.39849501 -0.02068080  0.26173015 -0.01141257
0.21203224 -0.00813028  0.18047769 -0.00609872  0.15307232 -0.00439072  0.13856170 -0.00351120
0.11601319 -0.00217031"
tDTable(14) = "000000016  0.71706417 -0.04209715  0.39748253 -0.02071012  0.26031962 -0.01143714
0.21043090 -0.00814963  0.17871892 -0.00611244  0.15111669 -0.00439052  0.13653337 -0.00352185
0.11348401 -0.00209587"
tDTable(15) = "000000017  0.71686413 -0.04210694  0.39659184 -0.02073963  0.25907851 -0.01146078
0.20902793 -0.00816941  0.17717916 -0.00612709  0.14942257 -0.00439920  0.13472445 -0.00352114
0.11143492 -0.00209112"
```

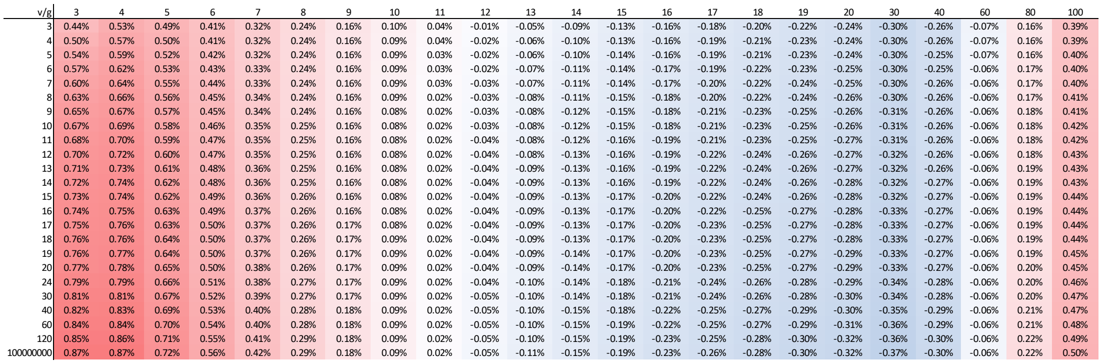
```
tDTable(16) = "000000018  0.71668529 -0.04211857  0.39580245 -0.02076887  0.25797963 -0.01148379
0.20778870 -0.00818907  0.17582027 -0.00614210  0.14792849 -0.00440883  0.13311054 -0.00351606
0.10950163 -0.00204780"
tDTable(17) = "000000019  0.71652541 -0.04213181  0.39509832 -0.02079750  0.25700056 -0.01150615
0.20668084 -0.00820680  0.17461200 -0.00615698  0.14660126 -0.00441895  0.13170389 -0.00352320
0.10803667 -0.00208734"
tDTable(18) = "000000020  0.71638132 -0.04214617  0.39446630 -0.02082529  0.25611886 -0.01152693
0.20567813 -0.00822131  0.17353095 -0.00617157  0.14541427 -0.00442918  0.13044657 -0.00353080
0.10684425 -0.00215181"
tDTable(19) = "000000024  0.71592629 -0.04220851  0.39247792 -0.02092678  0.25337841 -0.01160920
0.20254960 -0.00828078  0.17014647 -0.00622462  0.14170402 -0.00446851  0.12652064 -0.00356208
0.10297929 -0.00228433"
tDTable(20) = "000000030  0.71548581 -0.04230354  0.39050886 -0.02104877  0.25067306 -0.01170275
0.19949985 -0.00835910  0.16675491 -0.00627614  0.13806620 -0.00451714  0.12267891 -0.00360340
0.09854307 -0.00223936"
tDTable(21) = "000000040  0.71507852 -0.04244185  0.38856087 -0.02119161  0.24800445 -0.01180770
0.19648874 -0.00844366  0.16346955 -0.00634590  0.13451202 -0.00458544  0.11891984 -0.00365330
0.09441685 -0.00228192"
tDTable(22) = "000000060  0.71472243 -0.04263446  0.38663389 -0.02135459  0.24537091 -0.01192204
0.19352367 -0.00853648  0.16024078 -0.00642410  0.13091916 -0.00463190  0.11529640 -0.00373290
0.09040329 -0.00233387"
tDTable(23) = "000000120  0.71444644 -0.04289677  0.38473372 -0.02153718  0.24277101 -0.01204387
0.19060503 -0.00863703  0.15706958 -0.00651084  0.12747915 -0.00470721  0.11162556 -0.00379309
0.08649381 -0.00239241"
tDTable(24) = "100000000  0.71426748 -0.04323940  0.38283858 -0.02172935  0.24020454 -0.01217198
0.18773403 -0.00874542  0.15395724 -0.00660656  0.12411223 -0.00479232  0.10808061 -0.00386795
0.08267812 -0.00245567"""


For i = 1 To 24
    df(i) = Val(Left(tDTable(i), 9))
Next i

'If g=2 proceed with exact calculation and go to the end of the function.
If g = 2 Then
    y_temp = 1
    GoTo EndTDINV:
End If

'Set the alpha values corresponding to the eight tD tables.
alpha(1) = 0.5: alpha(2) = 0.25: alpha(3) = 0.1: alpha(4) = 0.05: alpha(5) = 0.025: alpha(6) =
0.01: alpha(7) = 0.005: alpha(8) = 0.001

'Select two lines from A coefficient table corresponding to df's higher and lower (v_low, v_high)
than the desired df (or v).
If v = 100000000 Or v > 100000000 Then
    v = 100000000: iMin = 23: iMax = 24
    GoTo BypassHiLo:
End If

For i = 2 To 24
    If (v = df(i - 1) Or v > df(i - 1)) And v < df(i) Then
        iMin = i - 1: iMax = i
    End If
Next i

BypassHiLo:

tDLine(1) = tDTable(iMin)
tDLine(2) = tDTable(iMax)
v_low = df(iMin)
v_high = df(iMax)

'After the above is done, you basically have a two-line text table that looks like this:

'df's     a=0.5   a=0.25  ... a=0.001
'-------------------------------------------------------
' v_low   A1 A2   A1 A2  ...  A1 A2   =tDLine(1)
'v_high   A1 A2   A1 A2  ...  A1 A2   =tDLine(2)
'-------------------------------------------------------
```

```
ReDim z(1 To 8)
ReDim x(1 To 2, 1 To 8)

For i = 1 To 8
    'Calculate y2low and y2high for each of the eight alpha's using the appropriate two A
constants.
    y2_low(i) = (1 + Val(Mid(tDLine(1), (i - 1) * 24 + 10, 12)) * Log(g - 1) + Val(Mid(tDLine(1),
(i - 1) * 24 + 22, 12)) * (Log(g - 1)) ^ 2) ^ 2
    y2_high(i) = (1 + Val(Mid(tDLine(2), (i - 1) * 24 + 10, 12)) * Log(g - 1) +
Val(Mid(tDLine(2), (i - 1) * 24 + 22, 12)) * (Log(g - 1)) ^ 2) ^ 2
    'Interpolate y2 from y2low and y2high and take the square root to find y for the desired v.
This will yield eight y-values, one for each alpha.'
    y(i) = Sqr(y2_low(i) + (1 / v - 1 / v_low) * (y2_high(i) - y2_low(i)) / (1 / v_high - 1 /
v_low))
    'Calculate x-values as x= -1/(1+1.5*NORMINV(1-alpha/2,0,1) and then x2-values. This will
yield eight x- and eight x2-values, one for each alpha.
    x(1, i) = -1 / (1 + 1.5 * Application.WorksheetFunction.NormInv(1 - alpha(i) / 2, 0, 1))
    x(2, i) = x(1, i) ^ 2
    'Calculate z-values as z=LN(y+r/v). This will yield eight z-values, one for each alpha.
    z(i) = Log(y(i) + g / v)
Next i

'Assume that the variable z is dependent on x through, 2nd polynomial, z=fn(x).

'Calculate a temporary x-value for the desired alpha and use the polynomial constants to find its
temporary z-value.
x_temp = -1 / (1 + 1.5 * Application.WorksheetFunction.NormInv(1 - a / 2, 0, 1))
z_temp = Application.WorksheetFunction.Index(Application.WorksheetFunction.LinEst(z, x, True), 3)
+ Application.WorksheetFunction.Index(Application.WorksheetFunction.LinEst(z, x, True), 2) *
x_temp + Application.WorksheetFunction.Index(Application.WorksheetFunction.LinEst(z, x, True), 1)
* x_temp ^ 2

'From the z-value, calculate the y-value.
y_temp = Exp(z_temp) - g / v

'Calculate the final tD-value from tD=y*TINV(alpha, v). Note that TINV is an Excel function.
EndTDINV:
TDINV = y_temp * Application.WorksheetFunction.TInv(a, v)

End Function
```

Figure S1. Excel spreadsheet with method suitable for Google Sheet implementation. *Right-click on the graph icon to Open File and save as separate file.*

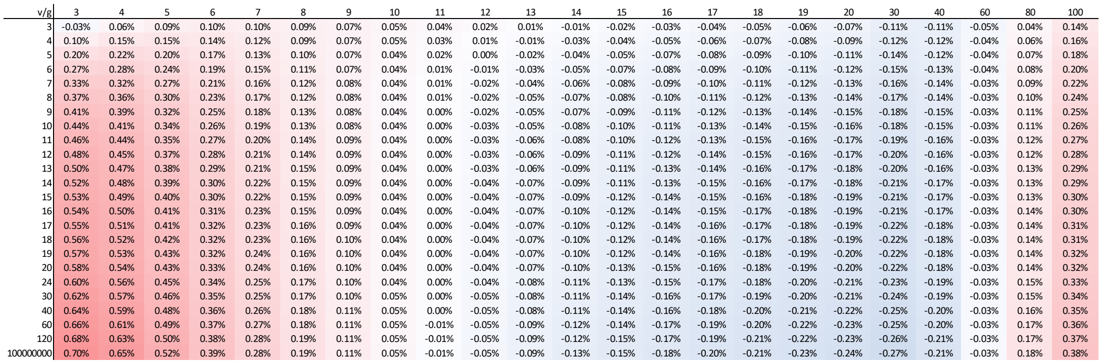


df (or v)=	19		tD=	**2.547827605**
g=	4		📊	
alpha=	0.05			
alpha >>	0.5		0.25	
v (df)	A1	A2	A1	A2
3	0.72397914	-0.04288019	0.46150139	-0.02255796
4	0.72302968	-0.04267121	0.44256976	-0.02158658
5	0.72214379	-0.04250446	0.43086420	-0.02109903
6	0.72133342	-0.04237675	0.42289904	-0.02083655
7	0.72061179	-0.04228144	0.41714139	-0.02069517
8	0.71997833	-0.04221184	0.41278848	-0.02062177
9	0.71942349	-0.04216181	0.40938449	-0.02058756
10	0.71893973	-0.04212728	0.40665749	-0.02057910
11	0.71851921	-0.04210515	0.40442553	-0.02058619
12	0.71814945	-0.04209195	0.40256735	-0.02060306
13	0.71782583	-0.04208613	0.40099769	-0.02062592
14	0.71753998	-0.04208571	0.39965537	-0.02065239
15	0.71728839	-0.04208987	0.39849501	-0.02068080
16	0.71706417	-0.04209715	0.39748253	-0.02071012
17	0.71686413	-0.04210694	0.39659184	-0.02073963
18	0.71668529	-0.04211857	0.39580245	-0.02076887
19	0.71652541	-0.04213181	0.39509832	-0.02079750



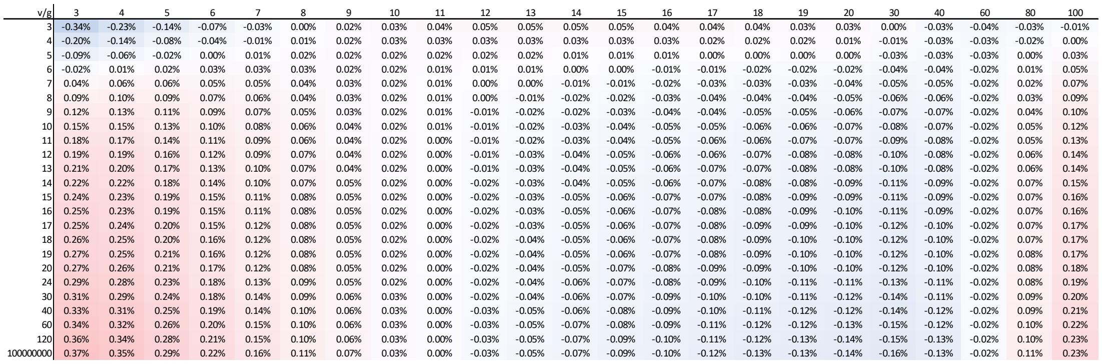
v/g	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	30	40	60	80	100
3	0.44%	0.53%	0.49%	0.41%	0.32%	0.24%	0.16%	0.10%	0.04%	-0.01%	-0.05%	-0.09%	-0.13%	-0.16%	-0.18%	-0.20%	-0.22%	-0.24%	-0.30%	-0.26%	-0.07%	0.16%	0.39%
4	0.50%	0.57%	0.50%	0.41%	0.32%	0.24%	0.16%	0.09%	0.04%	-0.02%	-0.06%	-0.10%	-0.13%	-0.16%	-0.19%	-0.21%	-0.23%	-0.24%	-0.30%	-0.26%	-0.07%	0.16%	0.39%
5	0.54%	0.59%	0.52%	0.42%	0.32%	0.24%	0.16%	0.09%	0.03%	-0.02%	-0.06%	-0.10%	-0.14%	-0.16%	-0.19%	-0.21%	-0.23%	-0.24%	-0.30%	-0.25%	-0.07%	0.16%	0.40%
6	0.57%	0.62%	0.53%	0.43%	0.33%	0.24%	0.16%	0.09%	0.03%	-0.02%	-0.07%	-0.11%	-0.14%	-0.17%	-0.19%	-0.22%	-0.23%	-0.25%	-0.30%	-0.25%	-0.06%	0.17%	0.40%
7	0.60%	0.64%	0.55%	0.44%	0.33%	0.24%	0.16%	0.09%	0.03%	-0.03%	-0.07%	-0.11%	-0.14%	-0.17%	-0.20%	-0.22%	-0.24%	-0.25%	-0.30%	-0.26%	-0.06%	0.17%	0.40%
8	0.63%	0.66%	0.56%	0.45%	0.34%	0.24%	0.16%	0.09%	0.02%	-0.03%	-0.08%	-0.11%	-0.15%	-0.18%	-0.20%	-0.22%	-0.24%	-0.26%	-0.30%	-0.26%	-0.06%	0.17%	0.41%
9	0.65%	0.67%	0.57%	0.45%	0.34%	0.24%	0.16%	0.08%	0.02%	-0.03%	-0.08%	-0.12%	-0.15%	-0.18%	-0.21%	-0.23%	-0.25%	-0.26%	-0.31%	-0.26%	-0.06%	0.18%	0.41%
10	0.67%	0.69%	0.58%	0.46%	0.35%	0.25%	0.16%	0.08%	0.02%	-0.03%	-0.08%	-0.12%	-0.15%	-0.18%	-0.21%	-0.23%	-0.25%	-0.26%	-0.31%	-0.26%	-0.06%	0.18%	0.42%
11	0.68%	0.70%	0.59%	0.47%	0.35%	0.25%	0.16%	0.08%	0.02%	-0.04%	-0.08%	-0.12%	-0.16%	-0.19%	-0.21%	-0.23%	-0.25%	-0.27%	-0.31%	-0.26%	-0.06%	0.18%	0.42%
12	0.70%	0.72%	0.60%	0.47%	0.35%	0.25%	0.16%	0.08%	0.02%	-0.04%	-0.08%	-0.13%	-0.16%	-0.19%	-0.22%	-0.24%	-0.26%	-0.27%	-0.32%	-0.26%	-0.06%	0.18%	0.43%
13	0.71%	0.73%	0.61%	0.48%	0.36%	0.25%	0.16%	0.08%	0.02%	-0.04%	-0.09%	-0.13%	-0.16%	-0.19%	-0.22%	-0.24%	-0.26%	-0.27%	-0.32%	-0.26%	-0.06%	0.19%	0.43%
14	0.72%	0.74%	0.62%	0.48%	0.36%	0.25%	0.16%	0.08%	0.02%	-0.04%	-0.09%	-0.13%	-0.16%	-0.19%	-0.22%	-0.24%	-0.26%	-0.28%	-0.32%	-0.27%	-0.06%	0.19%	0.43%
15	0.73%	0.74%	0.62%	0.49%	0.36%	0.26%	0.16%	0.08%	0.02%	-0.04%	-0.09%	-0.13%	-0.17%	-0.20%	-0.22%	-0.24%	-0.26%	-0.28%	-0.32%	-0.27%	-0.06%	0.19%	0.44%
16	0.74%	0.75%	0.63%	0.49%	0.37%	0.26%	0.16%	0.08%	0.02%	-0.04%	-0.09%	-0.13%	-0.17%	-0.20%	-0.22%	-0.25%	-0.27%	-0.28%	-0.33%	-0.27%	-0.06%	0.19%	0.44%
17	0.75%	0.76%	0.63%	0.50%	0.37%	0.26%	0.17%	0.08%	0.02%	-0.04%	-0.09%	-0.13%	-0.17%	-0.20%	-0.23%	-0.25%	-0.27%	-0.28%	-0.33%	-0.27%	-0.06%	0.19%	0.44%
18	0.76%	0.76%	0.64%	0.50%	0.37%	0.26%	0.17%	0.09%	0.02%	-0.04%	-0.09%	-0.13%	-0.17%	-0.20%	-0.23%	-0.25%	-0.27%	-0.28%	-0.33%	-0.27%	-0.06%	0.19%	0.44%
19	0.76%	0.77%	0.64%	0.50%	0.37%	0.26%	0.17%	0.09%	0.02%	-0.04%	-0.09%	-0.14%	-0.17%	-0.20%	-0.23%	-0.25%	-0.27%	-0.29%	-0.33%	-0.27%	-0.06%	0.19%	0.45%
20	0.77%	0.78%	0.65%	0.50%	0.38%	0.26%	0.17%	0.09%	0.02%	-0.04%	-0.09%	-0.14%	-0.17%	-0.20%	-0.23%	-0.25%	-0.27%	-0.29%	-0.33%	-0.27%	-0.06%	0.20%	0.45%
24	0.79%	0.79%	0.66%	0.51%	0.38%	0.27%	0.17%	0.09%	0.02%	-0.04%	-0.10%	-0.14%	-0.18%	-0.21%	-0.24%	-0.26%	-0.28%	-0.29%	-0.34%	-0.28%	-0.06%	0.20%	0.46%
30	0.81%	0.81%	0.67%	0.52%	0.39%	0.27%	0.17%	0.09%	0.02%	-0.05%	-0.10%	-0.14%	-0.18%	-0.21%	-0.24%	-0.26%	-0.28%	-0.30%	-0.34%	-0.28%	-0.06%	0.20%	0.47%
40	0.82%	0.83%	0.69%	0.53%	0.40%	0.28%	0.18%	0.09%	0.02%	-0.05%	-0.10%	-0.15%	-0.18%	-0.22%	-0.25%	-0.27%	-0.29%	-0.30%	-0.35%	-0.29%	-0.06%	0.21%	0.47%
60	0.84%	0.84%	0.70%	0.54%	0.40%	0.28%	0.18%	0.09%	0.02%	-0.05%	-0.10%	-0.15%	-0.19%	-0.22%	-0.25%	-0.27%	-0.29%	-0.31%	-0.36%	-0.29%	-0.06%	0.21%	0.48%
120	0.85%	0.86%	0.71%	0.55%	0.41%	0.29%	0.18%	0.09%	0.02%	-0.05%	-0.10%	-0.15%	-0.19%	-0.23%	-0.25%	-0.28%	-0.30%	-0.32%	-0.36%	-0.30%	-0.06%	0.22%	0.49%
100000000	0.87%	0.87%	0.72%	0.56%	0.42%	0.29%	0.18%	0.09%	0.02%	-0.05%	-0.11%	-0.15%	-0.19%	-0.23%	-0.26%	-0.28%	-0.30%	-0.32%	-0.37%	-0.30%	-0.06%	0.22%	0.50%

Figure S2. Error map of α=0.50, comparing the developed method with values from SAS Studio.



v/g	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	30	40	60	80	100
3	-0.03%	0.06%	0.09%	0.10%	0.10%	0.09%	0.07%	0.05%	0.04%	0.02%	0.01%	-0.01%	-0.02%	-0.03%	-0.04%	-0.05%	-0.06%	-0.07%	-0.11%	-0.11%	-0.05%	0.04%	0.14%
4	0.10%	0.15%	0.15%	0.14%	0.12%	0.09%	0.07%	0.05%	0.03%	0.01%	-0.01%	-0.03%	-0.04%	-0.05%	-0.06%	-0.07%	-0.08%	-0.09%	-0.12%	-0.12%	-0.04%	0.06%	0.16%
5	0.20%	0.22%	0.20%	0.17%	0.13%	0.10%	0.07%	0.04%	0.02%	0.00%	-0.02%	-0.04%	-0.05%	-0.07%	-0.08%	-0.09%	-0.10%	-0.11%	-0.14%	-0.12%	-0.04%	0.07%	0.18%
6	0.27%	0.28%	0.24%	0.19%	0.15%	0.11%	0.07%	0.04%	0.01%	-0.01%	-0.03%	-0.05%	-0.07%	-0.08%	-0.09%	-0.10%	-0.11%	-0.12%	-0.15%	-0.13%	-0.04%	0.08%	0.20%
7	0.33%	0.32%	0.27%	0.21%	0.16%	0.12%	0.08%	0.04%	0.01%	-0.02%	-0.04%	-0.06%	-0.08%	-0.09%	-0.10%	-0.11%	-0.12%	-0.13%	-0.16%	-0.14%	-0.03%	0.09%	0.22%
8	0.37%	0.36%	0.30%	0.23%	0.17%	0.12%	0.08%	0.04%	0.01%	-0.02%	-0.05%	-0.07%	-0.08%	-0.10%	-0.11%	-0.12%	-0.13%	-0.14%	-0.17%	-0.14%	-0.03%	0.10%	0.24%
9	0.41%	0.39%	0.32%	0.25%	0.18%	0.13%	0.08%	0.04%	0.00%	-0.02%	-0.05%	-0.07%	-0.09%	-0.11%	-0.12%	-0.13%	-0.14%	-0.15%	-0.18%	-0.15%	-0.03%	0.11%	0.25%
10	0.44%	0.41%	0.34%	0.26%	0.19%	0.13%	0.08%	0.04%	0.00%	-0.03%	-0.05%	-0.08%	-0.10%	-0.11%	-0.13%	-0.14%	-0.15%	-0.16%	-0.18%	-0.15%	-0.03%	0.11%	0.26%
11	0.46%	0.44%	0.35%	0.27%	0.20%	0.14%	0.09%	0.04%	0.00%	-0.03%	-0.06%	-0.08%	-0.10%	-0.12%	-0.13%	-0.15%	-0.16%	-0.17%	-0.19%	-0.16%	-0.03%	0.12%	0.27%
12	0.48%	0.45%	0.37%	0.28%	0.21%	0.14%	0.09%	0.04%	0.00%	-0.03%	-0.06%	-0.09%	-0.11%	-0.12%	-0.14%	-0.15%	-0.16%	-0.17%	-0.20%	-0.16%	-0.03%	0.12%	0.28%
13	0.50%	0.47%	0.38%	0.29%	0.21%	0.15%	0.09%	0.04%	0.00%	-0.03%	-0.06%	-0.09%	-0.11%	-0.13%	-0.14%	-0.16%	-0.17%	-0.18%	-0.20%	-0.16%	-0.03%	0.13%	0.29%
14	0.52%	0.48%	0.39%	0.30%	0.22%	0.15%	0.09%	0.04%	0.00%	-0.04%	-0.07%	-0.09%	-0.11%	-0.13%	-0.15%	-0.16%	-0.17%	-0.18%	-0.21%	-0.17%	-0.03%	0.13%	0.29%
15	0.53%	0.49%	0.40%	0.30%	0.22%	0.15%	0.09%	0.04%	0.00%	-0.04%	-0.07%	-0.09%	-0.12%	-0.14%	-0.15%	-0.16%	-0.18%	-0.19%	-0.21%	-0.17%	-0.03%	0.13%	0.30%
16	0.54%	0.50%	0.41%	0.31%	0.23%	0.15%	0.09%	0.04%	0.00%	-0.04%	-0.07%	-0.10%	-0.12%	-0.14%	-0.15%	-0.17%	-0.18%	-0.19%	-0.21%	-0.17%	-0.03%	0.14%	0.30%
17	0.55%	0.51%	0.41%	0.32%	0.23%	0.16%	0.09%	0.04%	0.00%	-0.04%	-0.07%	-0.10%	-0.12%	-0.14%	-0.16%	-0.17%	-0.18%	-0.19%	-0.22%	-0.18%	-0.03%	0.14%	0.31%
18	0.56%	0.52%	0.42%	0.32%	0.23%	0.16%	0.10%	0.04%	0.00%	-0.04%	-0.07%	-0.10%	-0.12%	-0.14%	-0.16%	-0.17%	-0.18%	-0.19%	-0.22%	-0.18%	-0.03%	0.14%	0.31%
19	0.57%	0.53%	0.43%	0.32%	0.24%	0.16%	0.10%	0.04%	0.00%	-0.04%	-0.07%	-0.10%	-0.12%	-0.14%	-0.16%	-0.18%	-0.19%	-0.20%	-0.22%	-0.18%	-0.03%	0.14%	0.32%
20	0.58%	0.54%	0.43%	0.33%	0.24%	0.16%	0.10%	0.04%	0.00%	-0.04%	-0.07%	-0.10%	-0.13%	-0.15%	-0.16%	-0.18%	-0.19%	-0.20%	-0.22%	-0.18%	-0.03%	0.14%	0.32%
24	0.60%	0.56%	0.45%	0.34%	0.25%	0.17%	0.10%	0.04%	0.00%	-0.04%	-0.08%	-0.11%	-0.13%	-0.15%	-0.17%	-0.18%	-0.20%	-0.21%	-0.23%	-0.19%	-0.03%	0.15%	0.33%
30	0.62%	0.57%	0.46%	0.35%	0.25%	0.17%	0.10%	0.05%	0.00%	-0.05%	-0.08%	-0.11%	-0.14%	-0.16%	-0.17%	-0.19%	-0.20%	-0.21%	-0.24%	-0.19%	-0.03%	0.15%	0.34%
40	0.64%	0.59%	0.48%	0.36%	0.26%	0.18%	0.11%	0.05%	0.00%	-0.05%	-0.08%	-0.11%	-0.14%	-0.16%	-0.18%	-0.20%	-0.21%	-0.22%	-0.25%	-0.20%	-0.03%	0.16%	0.35%
60	0.66%	0.61%	0.49%	0.37%	0.27%	0.18%	0.11%	0.05%	-0.01%	-0.05%	-0.09%	-0.12%	-0.14%	-0.17%	-0.19%	-0.20%	-0.22%	-0.23%	-0.25%	-0.20%	-0.03%	0.17%	0.36%
120	0.68%	0.63%	0.50%	0.38%	0.28%	0.19%	0.11%	0.05%	-0.01%	-0.05%	-0.09%	-0.12%	-0.15%	-0.17%	-0.19%	-0.21%	-0.22%	-0.23%	-0.26%	-0.21%	-0.03%	0.17%	0.37%
100000000	0.70%	0.65%	0.52%	0.39%	0.28%	0.19%	0.11%	0.05%	-0.01%	-0.05%	-0.09%	-0.13%	-0.15%	-0.18%	-0.20%	-0.21%	-0.23%	-0.24%	-0.27%	-0.21%	-0.03%	0.18%	0.38%

Figure S3. Error map of α=0.25.



v/g	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	30	40	60	80	100
3	-0.34%	-0.23%	-0.14%	-0.07%	-0.03%	0.00%	0.02%	0.03%	0.04%	0.05%	0.05%	0.05%	0.05%	0.04%	0.04%	0.04%	0.03%	0.03%	0.00%	-0.03%	-0.04%	-0.03%	-0.01%
4	-0.20%	-0.14%	-0.08%	-0.04%	-0.01%	0.01%	0.02%	0.03%	0.03%	0.03%	0.03%	0.03%	0.03%	0.03%	0.02%	0.02%	0.02%	0.01%	-0.01%	-0.03%	-0.03%	-0.02%	0.00%
5	-0.09%	-0.06%	-0.02%	0.00%	0.01%	0.02%	0.02%	0.02%	0.02%	0.02%	0.02%	0.01%	0.01%	0.01%	0.00%	0.00%	0.00%	0.00%	-0.03%	-0.03%	-0.03%	0.00%	0.03%
6	-0.02%	0.01%	0.02%	0.03%	0.03%	0.03%	0.02%	0.02%	0.01%	0.01%	0.01%	0.00%	0.00%	-0.01%	-0.01%	-0.02%	-0.02%	-0.02%	-0.04%	-0.04%	-0.02%	0.01%	0.05%
7	0.04%	0.06%	0.06%	0.05%	0.05%	0.04%	0.03%	0.02%	0.01%	0.00%	0.00%	-0.01%	-0.01%	-0.02%	-0.03%	-0.03%	-0.03%	-0.04%	-0.05%	-0.05%	-0.02%	0.02%	0.07%
8	0.09%	0.10%	0.09%	0.07%	0.06%	0.04%	0.03%	0.02%	0.01%	0.00%	-0.01%	-0.02%	-0.02%	-0.03%	-0.04%	-0.04%	-0.04%	-0.05%	-0.06%	-0.06%	-0.02%	0.03%	0.09%
9	0.12%	0.13%	0.11%	0.09%	0.07%	0.05%	0.03%	0.02%	0.01%	-0.01%	-0.02%	-0.02%	-0.03%	-0.04%	-0.04%	-0.05%	-0.05%	-0.06%	-0.07%	-0.07%	-0.02%	0.04%	0.10%
10	0.15%	0.15%	0.13%	0.10%	0.08%	0.06%	0.04%	0.02%	0.01%	-0.01%	-0.02%	-0.03%	-0.04%	-0.05%	-0.05%	-0.06%	-0.06%	-0.07%	-0.08%	-0.07%	-0.02%	0.05%	0.12%
11	0.18%	0.17%	0.14%	0.11%	0.09%	0.06%	0.04%	0.02%	0.00%	-0.01%	-0.02%	-0.03%	-0.04%	-0.05%	-0.06%	-0.06%	-0.07%	-0.07%	-0.09%	-0.08%	-0.02%	0.05%	0.13%
12	0.19%	0.19%	0.16%	0.12%	0.09%	0.07%	0.04%	0.02%	0.00%	-0.01%	-0.03%	-0.04%	-0.05%	-0.06%	-0.06%	-0.07%	-0.08%	-0.08%	-0.10%	-0.08%	-0.02%	0.06%	0.14%
13	0.21%	0.20%	0.17%	0.13%	0.10%	0.07%	0.04%	0.02%	0.00%	-0.01%	-0.03%	-0.04%	-0.05%	-0.06%	-0.07%	-0.07%	-0.08%	-0.08%	-0.10%	-0.08%	-0.02%	0.06%	0.14%
14	0.22%	0.22%	0.18%	0.14%	0.10%	0.07%	0.05%	0.02%	0.00%	-0.02%	-0.03%	-0.04%	-0.05%	-0.06%	-0.07%	-0.08%	-0.08%	-0.09%	-0.11%	-0.09%	-0.02%	0.07%	0.15%
15	0.24%	0.23%	0.19%	0.15%	0.11%	0.08%	0.05%	0.02%	0.00%	-0.02%	-0.03%	-0.05%	-0.06%	-0.07%	-0.07%	-0.08%	-0.09%	-0.09%	-0.11%	-0.09%	-0.02%	0.07%	0.16%
16	0.25%	0.23%	0.19%	0.15%	0.11%	0.08%	0.05%	0.02%	0.00%	-0.02%	-0.03%	-0.05%	-0.06%	-0.07%	-0.08%	-0.08%	-0.09%	-0.10%	-0.11%	-0.09%	-0.02%	0.07%	0.16%
17	0.25%	0.24%	0.20%	0.15%	0.12%	0.08%	0.05%	0.02%	0.00%	-0.02%	-0.03%	-0.05%	-0.06%	-0.07%	-0.08%	-0.09%	-0.09%	-0.10%	-0.12%	-0.10%	-0.02%	0.07%	0.17%
18	0.26%	0.25%	0.20%	0.16%	0.12%	0.08%	0.05%	0.02%	0.00%	-0.02%	-0.04%	-0.05%	-0.06%	-0.07%	-0.08%	-0.09%	-0.10%	-0.10%	-0.12%	-0.10%	-0.02%	0.07%	0.17%
19	0.27%	0.25%	0.21%	0.16%	0.12%	0.08%	0.05%	0.02%	0.00%	-0.02%	-0.04%	-0.05%	-0.06%	-0.07%	-0.08%	-0.09%	-0.10%	-0.10%	-0.12%	-0.10%	-0.02%	0.08%	0.17%
20	0.27%	0.26%	0.21%	0.17%	0.12%	0.08%	0.05%	0.02%	0.00%	-0.02%	-0.04%	-0.05%	-0.07%	-0.08%	-0.09%	-0.09%	-0.10%	-0.10%	-0.12%	-0.10%	-0.02%	0.08%	0.18%
24	0.29%	0.28%	0.23%	0.18%	0.13%	0.09%	0.05%	0.02%	0.00%	-0.02%	-0.04%	-0.06%	-0.07%	-0.08%	-0.09%	-0.10%	-0.11%	-0.11%	-0.13%	-0.11%	-0.02%	0.08%	0.19%
30	0.31%	0.29%	0.24%	0.18%	0.14%	0.09%	0.06%	0.03%	0.00%	-0.02%	-0.04%	-0.06%	-0.07%	-0.09%	-0.10%	-0.10%	-0.11%	-0.12%	-0.14%	-0.11%	-0.02%	0.09%	0.20%
40	0.33%	0.31%	0.25%	0.19%	0.14%	0.10%	0.06%	0.03%	0.00%	-0.03%	-0.05%	-0.06%	-0.08%	-0.09%	-0.10%	-0.11%	-0.12%	-0.12%	-0.14%	-0.12%	-0.02%	0.09%	0.21%
60	0.34%	0.32%	0.26%	0.20%	0.15%	0.10%	0.06%	0.03%	0.00%	-0.03%	-0.05%	-0.07%	-0.08%	-0.09%	-0.11%	-0.12%	-0.12%	-0.13%	-0.15%	-0.12%	-0.02%	0.10%	0.22%
120	0.36%	0.34%	0.28%	0.21%	0.15%	0.10%	0.06%	0.03%	0.00%	-0.03%	-0.05%	-0.07%	-0.09%	-0.10%	-0.11%	-0.12%	-0.13%	-0.14%	-0.15%	-0.13%	-0.02%	0.10%	0.23%
100000000	0.37%	0.35%	0.29%	0.22%	0.16%	0.11%	0.07%	0.03%	0.00%	-0.03%	-0.05%	-0.07%	-0.09%	-0.10%	-0.12%	-0.13%	-0.13%	-0.14%	-0.16%	-0.13%	-0.02%	0.11%	0.23%

Figure S4. Error map of α=0.10.

v/g	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	30	40	60	80	100
3	-0.50%	-0.39%	-0.26%	-0.16%	-0.09%	-0.04%	0.00%	0.02%	0.04%	0.06%	0.07%	0.08%	0.08%	0.09%	0.09%	0.09%	0.09%	0.09%	0.06%	0.02%	-0.04%	-0.08%	-0.10%
4	-0.37%	-0.31%	-0.22%	-0.14%	-0.09%	-0.04%	-0.01%	0.01%	0.03%	0.04%	0.06%	0.06%	0.07%	0.07%	0.08%	0.08%	0.08%	0.08%	0.06%	0.03%	-0.02%	-0.07%	-0.10%
5	-0.26%	-0.22%	-0.16%	-0.11%	-0.07%	-0.03%	-0.01%	0.01%	0.02%	0.03%	0.04%	0.05%	0.05%	0.06%	0.06%	0.06%	0.06%	0.06%	0.04%	0.02%	-0.02%	-0.05%	-0.07%
6	-0.19%	-0.16%	-0.11%	-0.08%	-0.05%	-0.02%	-0.01%	0.01%	0.02%	0.02%	0.03%	0.03%	0.04%	0.04%	0.04%	0.04%	0.04%	0.04%	0.03%	0.02%	-0.01%	-0.04%	-0.05%
7	-0.13%	-0.12%	-0.08%	-0.05%	-0.04%	-0.01%	0.00%	0.01%	0.01%	0.02%	0.02%	0.02%	0.03%	0.03%	0.03%	0.03%	0.03%	0.03%	0.02%	0.01%	-0.01%	-0.03%	-0.03%
8	-0.09%	-0.07%	-0.05%	-0.03%	-0.02%	-0.01%	0.00%	0.01%	0.01%	0.01%	0.01%	0.02%	0.02%	0.02%	0.02%	0.02%	0.02%	0.02%	0.01%	0.00%	-0.01%	-0.02%	-0.02%
9	-0.05%	-0.04%	-0.03%	-0.01%	-0.01%	0.00%	0.00%	0.01%	0.00%	0.01%	0.01%	0.01%	0.01%	0.01%	0.01%	0.01%	0.01%	0.01%	0.00%	-0.01%	-0.01%	-0.01%	0.00%
10	-0.03%	-0.02%	-0.01%	0.00%	0.00%	0.01%	0.01%	0.01%	0.01%	0.01%	0.01%	0.01%	0.00%	0.00%	0.00%	0.00%	0.00%	-0.01%	-0.01%	-0.01%	-0.01%	0.00%	0.01%
11	-0.01%	0.00%	0.01%	0.01%	0.01%	0.01%	0.01%	0.01%	0.01%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	-0.01%	-0.01%	-0.01%	-0.02%	-0.02%	-0.01%	0.01%	0.02%
12	0.01%	0.02%	0.02%	0.02%	0.02%	0.01%	0.01%	0.01%	0.00%	0.00%	0.00%	0.00%	0.00%	-0.01%	-0.01%	-0.01%	-0.01%	-0.01%	-0.02%	-0.02%	-0.01%	0.01%	0.03%
13	0.03%	0.03%	0.03%	0.03%	0.02%	0.02%	0.01%	0.01%	0.00%	0.00%	0.00%	-0.01%	-0.01%	-0.01%	-0.01%	-0.01%	-0.02%	-0.02%	-0.03%	-0.03%	-0.01%	0.01%	0.04%
14	0.04%	0.04%	0.04%	0.03%	0.03%	0.02%	0.01%	0.01%	0.00%	0.00%	0.00%	-0.01%	-0.01%	-0.01%	-0.02%	-0.02%	-0.02%	-0.02%	-0.03%	-0.03%	-0.01%	0.02%	0.04%
15	0.05%	0.05%	0.04%	0.04%	0.03%	0.02%	0.02%	0.01%	0.00%	0.00%	-0.01%	-0.01%	-0.01%	-0.02%	-0.02%	-0.02%	-0.02%	-0.03%	-0.03%	-0.03%	-0.01%	0.02%	0.05%
16	0.06%	0.06%	0.05%	0.04%	0.03%	0.02%	0.02%	0.01%	0.00%	0.00%	-0.01%	-0.01%	-0.02%	-0.02%	-0.02%	-0.02%	-0.03%	-0.03%	-0.04%	-0.03%	-0.01%	0.02%	0.05%
17	0.06%	0.06%	0.06%	0.05%	0.04%	0.03%	0.02%	0.01%	0.00%	0.00%	-0.01%	-0.01%	-0.02%	-0.02%	-0.02%	-0.03%	-0.03%	-0.03%	-0.04%	-0.04%	-0.01%	0.02%	0.06%
18	0.07%	0.07%	0.06%	0.05%	0.04%	0.03%	0.02%	0.01%	0.00%	0.00%	-0.01%	-0.01%	-0.02%	-0.02%	-0.03%	-0.03%	-0.03%	-0.03%	-0.04%	-0.04%	-0.01%	0.02%	0.06%
19	0.08%	0.08%	0.07%	0.05%	0.04%	0.03%	0.02%	0.01%	0.00%	0.00%	-0.01%	-0.02%	-0.02%	-0.02%	-0.03%	-0.03%	-0.03%	-0.04%	-0.04%	-0.04%	-0.01%	0.03%	0.06%
20	0.08%	0.08%	0.07%	0.06%	0.04%	0.03%	0.02%	0.01%	0.00%	0.00%	-0.01%	-0.02%	-0.02%	-0.03%	-0.03%	-0.03%	-0.04%	-0.04%	-0.05%	-0.04%	-0.01%	0.03%	0.07%
24	0.10%	0.10%	0.08%	0.06%	0.05%	0.03%	0.02%	0.01%	0.00%	-0.01%	-0.01%	-0.02%	-0.03%	-0.03%	-0.03%	-0.04%	-0.04%	-0.04%	-0.05%	-0.04%	-0.01%	0.03%	0.08%
30	0.11%	0.11%	0.09%	0.07%	0.05%	0.04%	0.02%	0.01%	0.00%	-0.01%	-0.02%	-0.02%	-0.03%	-0.03%	-0.04%	-0.04%	-0.05%	-0.05%	-0.06%	-0.05%	-0.01%	0.04%	0.09%
40	0.13%	0.12%	0.10%	0.08%	0.06%	0.04%	0.03%	0.01%	0.00%	-0.01%	-0.02%	-0.03%	-0.03%	-0.04%	-0.04%	-0.05%	-0.05%	-0.05%	-0.06%	-0.05%	-0.01%	0.04%	0.09%
60	0.14%	0.13%	0.11%	0.09%	0.06%	0.05%	0.03%	0.01%	0.00%	-0.01%	-0.02%	-0.03%	-0.04%	-0.04%	-0.05%	-0.05%	-0.06%	-0.06%	-0.07%	-0.06%	-0.01%	0.04%	0.10%
120	0.15%	0.15%	0.12%	0.09%	0.07%	0.05%	0.03%	0.01%	0.00%	-0.01%	-0.02%	-0.03%	-0.04%	-0.05%	-0.05%	-0.06%	-0.06%	-0.06%	-0.07%	-0.06%	-0.01%	0.05%	0.11%
100000000	0.17%	0.16%	0.13%	0.10%	0.08%	0.05%	0.03%	0.01%	0.00%	-0.01%	-0.02%	-0.03%	-0.04%	-0.05%	-0.06%	-0.06%	-0.06%	-0.07%	-0.08%	-0.07%	-0.01%	0.05%	0.12%

Figure S5. Error map of α=0.025.

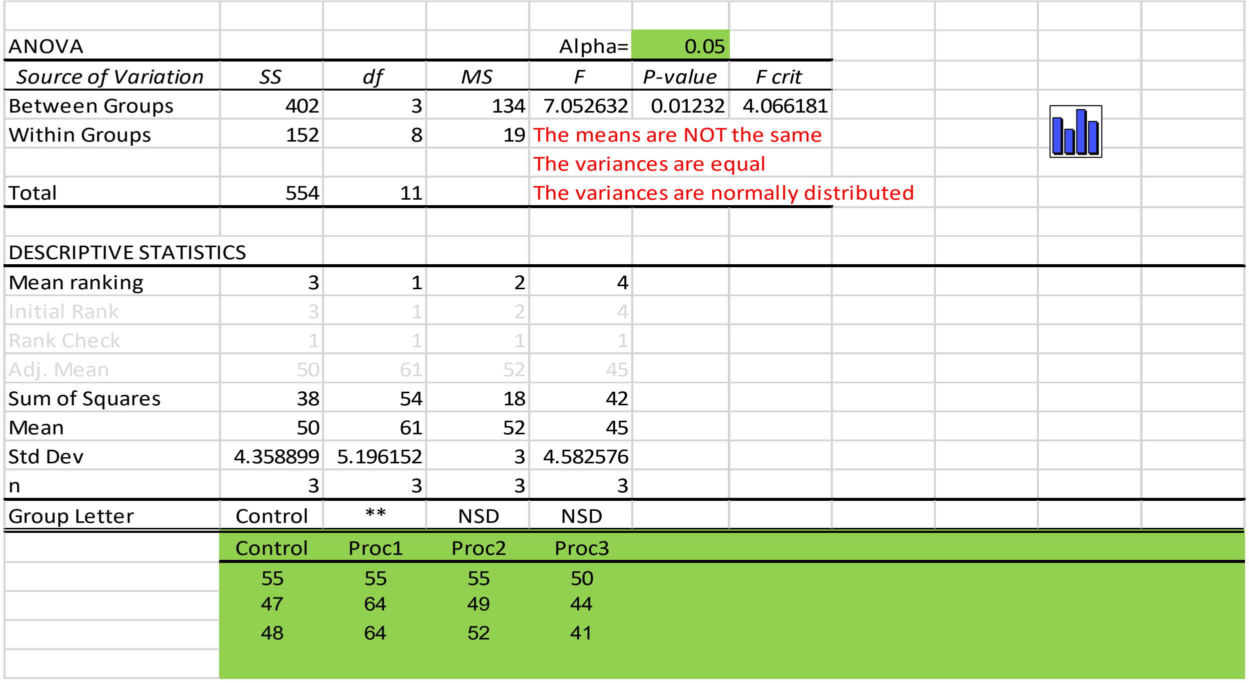
v/g	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	30	40	60	80	100
3	-0.53%	-0.44%	-0.28%	-0.18%	-0.11%	-0.06%	-0.01%	0.02%	0.04%	0.06%	0.07%	0.08%	0.09%	0.10%	0.10%	0.10%	0.10%	0.10%	0.08%	0.04%	-0.03%	-0.09%	-0.13%
4	-0.47%	-0.36%	-0.27%	-0.18%	-0.12%	-0.06%	-0.02%	0.01%	0.03%	0.05%	0.07%	0.08%	0.09%	0.09%	0.10%	0.10%	0.10%	0.11%	0.09%	0.05%	-0.02%	-0.09%	-0.14%
5	-0.34%	-0.29%	-0.21%	-0.15%	-0.09%	-0.05%	-0.02%	0.01%	0.03%	0.01%	0.05%	0.06%	0.06%	0.05%	0.09%	0.09%	0.10%	0.10%	0.09%	0.06%	-0.01%	-0.07%	-0.13%
6	-0.27%	-0.24%	-0.18%	-0.13%	-0.09%	-0.05%	-0.02%	0.00%	0.02%	0.03%	0.04%	0.05%	0.06%	0.06%	0.07%	0.07%	0.08%	0.08%	0.07%	0.05%	-0.01%	-0.06%	-0.11%
7	-0.20%	-0.19%	-0.14%	-0.10%	-0.07%	-0.04%	-0.02%	0.00%	0.01%	0.02%	0.03%	0.04%	0.05%	0.05%	0.06%	0.06%	0.06%	0.06%	0.06%	0.04%	-0.01%	-0.05%	-0.09%
8	-0.16%	-0.15%	-0.12%	-0.10%	-0.08%	-0.06%	-0.04%	0.01%	0.02%	0.03%	0.04%	0.04%	0.05%	0.05%	0.06%	0.06%	0.06%	0.06%	0.03%	0.02%	-0.01%	-0.04%	-0.07%
9	-0.14%	-0.11%	-0.08%	-0.06%	-0.05%	-0.04%	-0.03%	-0.02%	-0.01%	0.02%	0.03%	0.03%	0.04%	0.04%	0.04%	0.04%	0.04%	0.05%	0.04%	0.03%	-0.01%	-0.04%	-0.06%
10	-0.10%	-0.10%	-0.07%	-0.05%	-0.03%	-0.01%	-0.01%	0.00%	0.01%	0.01%	0.01%	0.01%	0.01%	0.01%	0.04%	0.04%	0.04%	0.04%	0.03%	0.02%	0.00%	-0.03%	-0.05%
11	-0.08%	-0.07%	-0.07%	-0.05%	-0.03%	-0.01%	0.00%	0.01%	0.01%	0.02%	0.02%	0.03%	0.00%	0.01%	0.03%	0.02%	0.02%	0.02%	0.03%	0.02%	0.00%	-0.02%	-0.04%
12	-0.06%	-0.05%	-0.04%	-0.02%	-0.04%	-0.03%	-0.01%	0.00%	0.00%	0.01%	0.01%	0.02%	0.02%	0.02%	0.02%	0.02%	0.03%	0.03%	0.02%	0.00%	0.00%	-0.02%	-0.03%
13	-0.05%	-0.04%	-0.03%	-0.02%	-0.01%	0.00%	0.00%	0.01%	-0.01%	-0.01%	0.00%	0.00%	0.01%	0.01%	0.01%	0.01%	0.02%	0.02%	0.02%	0.02%	0.00%	-0.02%	-0.01%
14	-0.04%	-0.04%	-0.03%	-0.02%	-0.01%	0.00%	0.00%	0.01%	0.01%	0.01%	0.01%	0.01%	0.01%	0.02%	0.02%	-0.01%	-0.01%	0.00%	0.00%	0.00%	0.00%	0.00%	-0.01%
15	-0.03%	-0.03%	-0.02%	-0.01%	-0.01%	0.00%	0.00%	0.00%	0.00%	0.01%	0.01%	0.01%	0.01%	0.01%	0.01%	0.01%	0.01%	0.01%	-0.01%	-0.01%	-0.01%	0.00%	0.00%
16	-0.03%	-0.02%	-0.02%	-0.01%	-0.01%	0.00%	0.00%	0.00%	0.00%	0.00%	0.01%	0.01%	0.01%	0.01%	0.01%	0.01%	0.01%	0.01%	0.01%	0.00%	-0.01%	-0.01%	0.00%
17	-0.02%	-0.02%	-0.01%	-0.01%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	-0.01%
18	-0.01%	-0.01%	-0.01%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%
19	-0.01%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	-0.01%	0.00%	0.00%	0.01%
20	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	-0.01%	-0.01%	0.00%	0.00%	0.01%
24	0.01%	0.01%	0.01%	0.01%	0.01%	0.01%	0.01%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	-0.01%	-0.01%	-0.01%	-0.01%	-0.01%	-0.01%	-0.01%	0.01%	0.02%
30	0.03%	0.03%	0.02%	0.02%	0.02%	0.01%	0.01%	0.00%	0.00%	0.00%	0.00%	-0.01%	-0.01%	-0.01%	-0.01%	-0.01%	-0.01%	-0.01%	-0.02%	-0.02%	-0.01%	0.01%	0.03%
40	0.04%	0.04%	0.03%	0.03%	0.02%	0.01%	0.01%	0.01%	0.00%	0.00%	-0.01%	-0.01%	-0.01%	-0.01%	-0.01%	-0.02%	-0.02%	-0.02%	-0.02%	-0.02%	-0.01%	0.01%	0.04%
60	0.05%	0.05%	0.04%	0.03%	0.02%	0.02%	0.01%	0.00%	0.00%	0.00%	-0.01%	-0.01%	-0.01%	-0.02%	-0.02%	-0.02%	-0.02%	-0.02%	-0.02%	-0.02%	-0.01%	0.01%	0.04%
120	0.06%	0.06%	0.05%	0.05%	0.04%	0.03%	0.02%	0.02%	0.01%	-0.01%	-0.02%	-0.02%	-0.02%	-0.03%	-0.03%	-0.03%	-0.03%	-0.03%	-0.04%	-0.03%	-0.01%	0.02%	0.06%
100000000	0.07%	0.07%	0.06%	0.05%	0.04%	0.03%	0.03%	0.00%	0.00%	-0.01%	-0.02%	-0.02%	-0.02%	-0.03%	-0.03%	-0.03%	-0.03%	-0.04%	-0.04%	-0.03%	-0.01%	0.03%	0.06%

Figure S6. Error map of α=0.005.

v/g	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	30	40	60	80	100
3	-0.60%	-0.46%	-0.31%	-0.20%	-0.12%	-0.06%	-0.02%	0.02%	0.05%	0.07%	0.08%	0.09%	0.10%	0.11%	0.11%	0.11%	0.11%	0.11%	0.09%	0.05%	-0.03%	-0.10%	-0.15%
4	-0.57%	-0.45%	-0.30%	-0.18%	-0.09%	-0.05%	-0.01%	0.02%	0.04%	0.06%	0.07%	0.08%	0.09%	0.10%	0.10%	0.11%	0.11%	0.11%	0.09%	0.06%	-0.02%	-0.09%	-0.15%
5	-0.29%	-0.29%	-0.23%	-0.17%	-0.12%	-0.07%	-0.04%	-0.01%	0.02%	0.03%	0.05%	0.06%	0.07%	0.08%	0.09%	0.09%	0.10%	0.10%	0.10%	0.07%	-0.01%	-0.08%	-0.14%
6	-0.24%	-0.19%	-0.12%	-0.21%	-0.10%	-0.01%	0.05%	0.05%	0.04%	0.04%	0.04%	0.04%	0.03%	0.03%	0.03%	0.02%	0.02%	0.01%	0.14%	0.10%	0.02%	-0.06%	-0.13%
7	-0.26%	-0.24%	-0.18%	-0.14%	-0.09%	-0.06%	-0.03%	-0.01%	0.01%	0.03%	0.05%	0.06%	0.07%	0.08%	0.08%	0.09%	0.09%	0.10%	-0.02%	0.05%	0.03%	-0.06%	-0.13%
8	-0.20%	-0.18%	-0.14%	-0.11%	-0.07%	-0.05%	-0.02%	-0.01%	0.01%	0.02%	0.03%	0.04%	0.05%	0.05%	0.06%	0.06%	0.06%	0.07%	0.07%	0.05%	0.00%	-0.05%	-0.10%
9	-0.16%	-0.15%	-0.12%	-0.09%	-0.06%	-0.04%	-0.02%	-0.01%	0.01%	0.02%	0.03%	0.03%	0.04%	0.04%	0.05%	0.05%	0.05%	0.06%	0.06%	0.04%	0.00%	-0.04%	-0.08%
10	-0.12%	-0.23%	-0.09%	-0.07%	-0.05%	-0.03%	-0.01%	0.00%	0.01%	0.02%	0.03%	0.03%	0.04%	0.04%	0.04%	0.05%	0.05%	0.05%	0.05%	0.03%	0.00%	-0.04%	-0.08%
11	-0.11%	-0.09%	-0.08%	-0.15%	-0.03%	-0.02%	-0.01%	0.00%	0.01%	0.02%	0.02%	0.03%	0.03%	0.03%	0.04%	0.04%	0.04%	0.04%	0.04%	0.03%	0.00%	-0.03%	-0.06%
12	-0.11%	-0.09%	-0.06%	-0.04%	-0.02%	-0.05%	-0.06%	0.01%	0.01%	0.02%	0.02%	0.03%	0.03%	0.03%	0.03%	0.03%	0.03%	0.03%	0.03%	0.02%	0.00%	-0.03%	-0.05%
13	-0.12%	-0.09%	-0.06%	-0.04%	-0.02%	0.00%	0.01%	0.02%	0.03%	-0.02%	-0.05%	0.00%	-0.03%	0.04%	0.04%	0.04%	0.04%	0.04%	0.04%	0.02%	0.00%	-0.03%	-0.05%
14	-0.14%	-0.11%	-0.08%	-0.05%	-0.03%	-0.02%	0.00%	0.01%	0.01%	0.02%	0.03%	0.03%	0.04%	0.04%	0.04%	0.05%	-0.03%	0.01%	0.04%	0.02%	0.00%	-0.03%	-0.05%
15	-0.14%	-0.11%	-0.08%	-0.06%	-0.04%	-0.02%	-0.01%	0.00%	0.01%	0.02%	0.02%	0.03%	0.03%	0.03%	0.04%	0.04%	0.04%	0.04%	0.05%	-0.01%	0.00%	-0.03%	-0.05%
16	-0.15%	-0.11%	-0.08%	-0.06%	-0.04%	-0.02%	-0.01%	0.00%	0.01%	0.01%	0.02%	0.02%	0.03%	0.03%	0.03%	0.04%	0.04%	0.04%	0.04%	0.03%	-0.01%	-0.03%	-0.06%
17	-0.03%	-0.02%	-0.09%	-0.07%	-0.05%	-0.03%	-0.02%	-0.01%	0.00%	0.01%	0.01%	0.02%	0.02%	0.02%	0.03%	0.03%	0.03%	0.03%	0.04%	0.03%	0.01%	-0.01%	-0.07%
18	-0.01%	0.00%	0.02%	-0.07%	-0.05%	-0.04%	-0.02%	-0.01%	-0.01%	0.00%	0.01%	0.01%	0.01%	0.02%	0.02%	0.02%	0.02%	0.03%	0.03%	0.02%	0.00%	-0.02%	-0.03%
19	-0.01%	0.00%	0.01%	0.02%	0.03%	0.04%	-0.05%	-0.04%	-0.03%	-0.02%	-0.01%	-0.01%	0.00%	0.00%	0.00%	0.01%	0.01%	0.01%	0.02%	0.02%	0.01%	0.00%	-0.02%
20	-0.02%	-0.01%	0.00%	0.01%	0.02%	0.03%	0.03%	0.04%	0.04%	0.04%	-0.04%	-0.04%	-0.03%	-0.03%	-0.02%	-0.02%	-0.02%	-0.01%	0.00%	0.01%	0.01%	0.01%	0.00%
24	-0.03%	-0.03%	-0.03%	-0.02%	-0.02%	-0.01%	-0.01%	-0.01%	0.00%	0.00%	0.00%	0.00%	0.01%	0.01%	0.01%	0.01%	0.01%	0.01%	0.02%	0.03%	0.03%	-0.04%	-0.03%
30	-0.01%	-0.01%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%
40	0.01%	0.01%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	-0.01%	0.00%	0.00%	0.01%
60	0.02%	0.02%	0.01%	0.01%	0.01%	0.01%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	-0.01%	-0.01%	-0.01%	-0.01%	-0.01%	-0.01%	-0.01%	0.00%	0.01%	0.02%
120	0.03%	0.03%	0.02%	0.02%	0.01%	0.01%	0.01%	0.00%	0.00%	0.00%	0.00%	-0.01%	-0.01%	-0.01%	-0.01%	-0.01%	-0.01%	-0.01%	-0.02%	-0.01%	0.00%	0.01%	0.02%
100000000	0.03%	0.03%	0.03%	0.02%	0.02%	0.01%	0.01%	0.00%	0.00%	0.00%	-0.01%	-0.01%	-0.01%	-0.01%	-0.01%	-0.01%	-0.02%	-0.02%	-0.02%	-0.02%	0.00%	0.01%	0.03%

Figure S7. Error map of α=0.001.

Figure S8. Example Excel workbook of Dunnett's (and other) test. *Right-click on the graph icon to Open File and save as separate file.*



ANOVA				Alpha=	0.05	
Source of Variation	*SS*	*df*	*MS*	*F*	*P-value*	*F crit*
Between Groups	402	3	134	7.052632	0.01232	4.066181
Within Groups	152	8	19	The means are NOT the same		
				The variances are equal		
Total	554	11		The variances are normally distributed		
DESCRIPTIVE STATISTICS						
Mean ranking	3	1	2	4		
Initial Rank	3	1	2	4		
Rank Check	1	1	1	1		
Adj. Mean	50	61	52	45		
Sum of Squares	38	54	18	42		
Mean	50	61	52	45		
Std Dev	4.358899	5.196152	3	4.582576		
n	3	3	3	3		
Group Letter	Control	**	NSD	NSD		
	Control	Proc1	Proc2	Proc3		
	55	55	55	50		
	47	64	49	44		
	48	64	52	41		